Electro-Magnetic Transients

Power Technology Course – Unit 9
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- Time Functions
- First Order Transients

Class 2
- Sine & Cosine Functions
- Capacitor Bank Opening
- Second Order Transients
- Damping

Class 3
- Traveling Waves
- Reflection Coefficients
- EMTP Simulation Results

Class 4
- Lightning Surges
- Shielding Models
- Overhead Line Lightning Performance
Tab 1 - Time Functions & Transients

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Steady state analysis assumes perfect cosine waves that do not change over time.

Transient analysis considers many types of time varying functions:

- Sine & Cosine waves
- Unit Step function
- Exponential functions
- Surge functions
Sources of Transients

- Switching circuits
- Faults
- Lightning
Impacted by Transients

- Insulation Levels
- Surge Arresters
- Relaying
- Circuit Breakers
- Equipment Design
- System Operation
- Other
Basic Circuit Elements

- First and second order electrical systems are based upon linear elements of
  - Voltage and Current Sources
  - Resistance
  - Inductance
  - Capacitance

- A solid understanding of the fundamental physical behavior of these elements is KEY to the understanding of electromagnetic transients.
Ideal Sources

- **Ideal Voltage Source**
  - Internal Z = 0
  - The external circuit does not affect the voltage.
  - can supply an infinite amount
    - current
    - power

- **Ideal Current Source**
  - Internal Y = 0
  - The external circuit does not affect the current.
  - can supply an infinite amount
    - voltage
    - power
Resistance (Ohm)

\[ i = \frac{v}{R} \]

Dissipates power (Watts):

\[ P = i^2R \]

\[ P = \frac{V^2}{R} \]

Dissipates energy (Joules):

\[ E = P \cdot t \]
### Inductance (Henry) and Capacitance (Farad)

<table>
<thead>
<tr>
<th><strong>L - Inductance</strong></th>
<th><strong>C - Capacitance</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Stores Energy:</td>
<td>Magnetic Fields</td>
</tr>
<tr>
<td>Energy Stored:</td>
<td>Electric Fields</td>
</tr>
<tr>
<td>( \frac{1}{2} L i^2 )</td>
<td>( \frac{1}{2} C v^2 )</td>
</tr>
<tr>
<td>( v = -\frac{d\phi}{dt} )</td>
<td>( i = \frac{dQ}{dt} )</td>
</tr>
<tr>
<td>( N\phi = \lambda = Li )</td>
<td>( Q = Cv )</td>
</tr>
<tr>
<td>Differential eq:</td>
<td></td>
</tr>
<tr>
<td>( v = L \frac{di}{dt} )</td>
<td>( i = C \frac{dv}{dt} )</td>
</tr>
<tr>
<td>for</td>
<td></td>
</tr>
<tr>
<td>( v = \omega LI \cos \omega t )</td>
<td>( v = V \sin \omega t )</td>
</tr>
<tr>
<td>Impedance:</td>
<td>( X_L = \omega L )</td>
</tr>
<tr>
<td>Ohm’s Law:</td>
<td>( X_C = 1/\omega C )</td>
</tr>
<tr>
<td>( v = X_L I \cos \omega t )</td>
<td>( i = \frac{V}{X_C} \cos \omega t )</td>
</tr>
<tr>
<td>Integral eq.:</td>
<td>( i = \frac{1}{L} \int_0^t v(t)dt + i(0) )</td>
</tr>
<tr>
<td></td>
<td>( v = \frac{1}{C} \int_0^t i(t)dt + v(0) )</td>
</tr>
</tbody>
</table>
Single Circuit Ideal Element
Response to a Step Source

- **R**
  - Voltage: 
  - Current: 

- **L**
  - Voltage: 
  - Current: 

- **C**
  - Voltage:
Unit Step Function

- Is equal to 1 for all positive values of the independent variable (term between parenthesis)
  
  \[ u(t) = 1 \quad t > 0 \]

- Is equal to zero for all negative values of the independent variable (term between parenthesis)
  
  \[ u(t) = 0 \quad t < 0 \]
In general, the start of a step function can be delayed by $t_0$ seconds and its magnitude adjusted to $A$

$$Y = Au (t-t_0)$$
The function $Y$ above can be expressed as the sum of two unit step functions:

First starts at $t=2$ with a magnitude $+5$
Second starts at $t=4$ with a magnitude $-5$

$$Y = 5 \ u(t-2) - 5 \ u(t-4)$$
A first order circuit has only 1 element that stores energy, either:

1 inductor (L)

or

1 capacitor (C)

RL or RC circuits

The circuit response is an exponential wave with a time constant $\tau$ determined by the values of

$R \& L$

or

$R \& C$
The Exponential Function

\[ \text{slope} = \frac{1}{\tau} 1 - e^{-\frac{t}{\tau}} \]

\[ \begin{array}{ccc}
0 & 1.000 & 0.000 \\
1 & 0.368 & 0.632 \\
2 & 0.135 & 0.865 \\
3 & 0.050 & 0.950 \\
4 & 0.018 & 0.982 \\
5 & 0.007 & 0.993 \\
\end{array} \]
Points on the Exponential Curve

If you know any 2 points, you can determine the time constant and any other point!

\[ A(1) = A(0) e^{\frac{t_1 - t_0}{\tau}} \]

\[ A(2) = A(0) e^{\frac{t_2 - t_0}{\tau}} = A(1) e^{\frac{t_2 - t_1}{\tau}} \]
1) Find initial value: $A(0^+)$

2) Find final (steady state) value: $A(\infty)$

3) Calculate the time constant: 
   
   $\tau = \frac{L'}{R'}$ or $\tau = R' C'$

4) The time response is:

   $$A(t) = A(0^+) e^{-\frac{t}{\tau}} + A(\infty) [1 - e^{-\frac{t}{\tau}}]$$
   
   or
   
   $$A(t) = \left[ A(0^+) - A(\infty) \right] e^{-\frac{t}{\tau}} + A(\infty)$$
Step Response Values In Damped Circuits

**L**
- \( V(0^+) = ? \)
- \( i(0^+) = i(0^-) \)
- \( X(0^+) = \infty \)
- \( V(\infty) = 0 \)
- \( i(\infty) = ? \)
- \( X(\infty) = 0 \)

? = to be determined

**C**
- \( V(0^+) = V(0^-) \)
- \( i(0^+) = ? \)
- \( X(0^+) = 0 \)
- \( V(\infty) = ? \)
- \( i(\infty) = 0 \)
- \( X(\infty) = \infty \)
Step Response of 1st Order Circuits

- There are only 4 basic 1st order circuits:
  - Series RL with a step voltage source
  - Series RC with a step voltage source
  - Parallel RL with a step current source
  - Parallel RC with a step current source

- The solution to each are found on the following pages.
Series RL With a Step Voltage Source

\[
\tau = \frac{L}{R}
\]

\[
i = \frac{V}{R} \left(1 - e^{-\frac{t}{\tau}}\right)
\]

\[
V_R = V \left(1 - e^{-\frac{t}{\tau}}\right)
\]

\[
V_L = V e^{-\frac{t}{\tau}}
\]
Parallel RL With a Step Current Source

\[ \tau = \frac{L}{R} \]

\[ i_R = I e^{-\frac{t}{\tau}} \]

\[ i_L = I (1 - e^{-\frac{t}{\tau}}) \]

\[ V = IR e^{-\frac{t}{\tau}} \]
Series RC With a Step Voltage Source

\[ \tau = RC \]

\[ i = \left( \frac{V - V_C(0)}{R} \right) e^{-\frac{t}{\tau}} \]

\[ V_R = [V - V_C(0)] e^{-\frac{t}{\tau}} \]

\[ V_C = V (1 - e^{-\frac{t}{\tau}}) + V_C(0) e^{-\frac{t}{\tau}} \]
Parallel RC With a Step Current Source

\[ \tau = RC \]
\[ i_R = I (1 - e^{-\frac{t}{\tau}}) \]
\[ i_C = I e^{-\frac{t}{\tau}} \]
\[ V = I R (1 - e^{-\frac{t}{\tau}}) \]
Tab 2 - Sine and Cosine Transients

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Sine and Cosine Functions

\[ V_{LL} = \sqrt{3} \ V_{LN} \quad V_{\text{peak}} = \sqrt{2} \ V_{\text{rms}} \quad V_{LN \text{ peak}} = \sqrt{\frac{2}{3}} \ V_{LL \text{ rms}} \]
Waveform Digitization

10 degrees between points

30 degrees between points
Poor Waveform Digitization

75 degrees between points

200 degrees between points
Source Voltage & Load Voltage at Various Closing Angles

\[ V_{\text{load}} = 1 \sin(\omega t) \cdot u(t-t_0) \]
Resistor Voltage, Current, Power & Energy

\[ v(t) = V \cos \omega t \]
\[ i(t) = \frac{V}{R} \cos \omega t \]
\[ p(t) = \frac{V^2}{2R} (1 + \cos 2\omega t) \]
Source Voltage & Resistor Current
for Various Closing Angles

\[ I = \frac{1\sin(\omega t) \cdot u(t-t_0)}{R} \]

\[ R = \text{_____} \]
Inductor Voltage, Current, Power & Energy

\[ V(t) = V \cos \omega t \]
\[ i(t) = \frac{V}{X_L} \sin \omega t = \frac{V}{X_L} \cos(\omega t - 90) \]
\[ p(t) = \frac{V^2}{2X_L} \sin 2\omega t \]
Source Voltage & Inductor Current for Various Closing Angles

\[ I = 1 \left[ \sin(\omega t - \theta_z) - \sin(\omega t_0 - \theta_z) \right] \cdot u(t-t_0)/X_L \]

\[ \theta_z = \pi/2 \]

\[ X_L = \quad \]

1 V

\[ V_{\text{source}} \]

\[ I_{\text{inductor}} \]
Source Voltage & RL Current for Various Closing Angles

\[ I = I_0 \left[ \sin(\omega t - \theta_z) \cdot \sin(\omega t_0 - \theta_z) e^{(t-t_0)/\tau} \right] \cdot u(t-t_0)/Z \]

\[ \theta_z = \tan^{-1}(X_L/R) \]

\[ \tau = L/R = X_L/\omega R \]

\[ Z = \sqrt{X_L^2 + R^2} \]
Capacitor Voltage, Current, Power & Energy

\[ i(t) = I \cos \omega t \]
\[ v(t) = X_c I \sin \omega t = X_c I \cos(\omega t - 90) \]
\[ p(t) = \frac{X_c I^2}{2} \sin 2\omega t \]
Source Voltage & RC Current for Various Closing Angles

\[ I = I_s \left[ \sin(\omega t + \theta_z) + \frac{Z \sin(\omega t_0) \sin(\omega t + \theta_z)}{R} \right] e^{-\frac{(t-t_0)}{\tau}} u(t-t_0) / Z \]

\[ \theta_z = \tan^{-1}(X_C/R) \]

\[ \tau = RC = R/\omega X_C \]

\[ Z = \sqrt{X_C^2 + R^2} \]
Opening of a Capacitor Bank

- Open a capacitor at a current zero leaves a 1.0 pu trapped charge on the capacitor.
- A 2.0 pu peak voltage appears across the switch.
Capacitor Bank Opening Problem

- A capacitor bank rated 50 MVAR (3 phase) at 138 kV (rms LL) opens when the system is operated at 1.05 pu.
- The bank is solidly grounded.

  - Find the voltage trapped on the capacitor
  - Find the peak voltage across the switch
  - Find the capacitance of one phase
Second Order Circuits

- A second order circuit has 2 elements that store energy:
  - 1 inductor
  - 1 capacitor

- LC or RLC circuits

- The voltages and currents are (damped) sine/cosine waves at a ‘natural’ frequency determined by the values of L & C
Series LC Circuit

Undamped response to a step change in voltage:
Series LC Circuit (continued)

\[ i(t) = \left[ \frac{V - V_C(0)}{Z_S} \right] \sin(\omega_n t) \]

\[ V_L(t) = L (\frac{di}{dt}) = L \left[ (V - V_C(0)) / Z_S \right] \omega_n \cos(\omega_n t) \]

\[ \omega_n = \frac{1}{\sqrt{LC}} \quad \text{(rad/sec)} \]

\[ Z_S = \sqrt{\frac{L}{C}} \quad \text{(\Omega)} \]

\[ V_L(t) = \left[ (V - V_C(0)) \right] \cos \omega_n t \]

\[ V_C(t) = V - V_L(t) \]

\[ v_c(t) = V \left[ 1 - \cos(\omega_n t) \right] + v_c(0) \cos(\omega_n t) \]
Energize Capacitor Bank - No Initial Charge

\[ V_C = V - V_L \]

\[ V_C(0^+) = 0 \]
Energize Capacitor Bank
with Various Initial Charges

The peak of the overvoltage depends upon the initial charge $V_C(0)$. 

\[ \omega_N \cdot t \]

\[ V_L \]

\[ V_C(0) \]
Damping

- The energy in the transient response never dissipates in an ideal LC circuit; it oscillates between the L & C.

- Resistance:
  - dampens transients
  - takes energy out of the transient response
  - reduces the duration of the transient response
  - reduces overshoots
The damping ratio $\zeta$ is:

- for series RLC circuits
  \[
  \zeta = \frac{R}{2Z_s}
  \]

- for parallel RLC circuits
  \[
  \zeta = \frac{Z_s}{2R}
  \]

- Calculate the resistance $R$ at $\omega_n$
- High frequency power system transients are typically underdamped:
  \[0.0 < \zeta < 0.5\]
- For critically damped circuits $\zeta = 1$
Capacitor Voltages

Damping ratio \( V_{\text{peak}} \)

- 0.0: 2.00
- 0.05: 1.85
- 0.10: 1.73
- 0.20: 1.53
- 0.40: 1.25
- 0.60: 1.09

\( \omega_N \cdot t \)
Inductor Voltages

Damping ratio

0.00
0.05
0.10
0.20
0.40
0.60

\[
\omega_N t
\]

\[
[V]
\]
Energize Capacitor Bank - Voltages

The first peak voltage is almost the same

\[ V_{\text{cap}} \text{ [DC source]} \]

\[ V_{\text{cap}} \text{ [AC source]} \]

AC source voltage

Siemens Industry, Inc., Siemens Power Technologies International
The first peak voltage is practically the same.
Capacitor Bank Outrush Currents Problem

Your company has a 20 MVAR (3 phase) shunt capacitor bank (grounded wye) in a 115 kV substation.

You have been assigned to determine the outrush currents from the capacitor bank from single line to ground faults in the substation.

Assume a loop inductance of 0.1 mH.

Analyze as a single phase problem without damping, and calculate:

- the magnitude of the peak fault current,
- the frequency of the current,
- the time to the first peak.
Solution to Capacitor Bank Outrush Currents Problem

\[ C = \frac{Q}{\omega V^2} = \frac{20.6E6}{377(115000)^2} = 4 \ \mu F \]

given inductance: \[ L = 0.1 \ \text{mH} \]

surge impedance: \[ Z_c = \sqrt{L/C} = 5 \ \text{ohms} \]

natural frequency: \[ \omega_0 = \frac{1}{\sqrt{LC}} = 50,000 \ \text{radians/sec} \]

\[ f_0 = \frac{\omega_0}{2\pi} = 7,958 \ \text{Hz} \]

voltage: \[ V = 0 \ [\text{no voltage source}] \]

\[ V_c(0) = 115 \frac{\sqrt{2}}{\sqrt{3}} = 93.9 \ \text{kV} \]

current: \[ i(t) = \frac{V_c(0)}{Z_c} \sin(\omega_0 t) \quad \text{for } t > 0 \]

\[ i(t) = 18.8 \sin(50000t) \ \text{kA} \quad \text{for } t > 0 \]

peak current: \[ 18.8 \ \text{kA} \]

time to peak current: \[ \omega_0 t_p = \frac{\pi}{2} \quad t_p = 31.4 \ \mu s \]
Tab 3 - Traveling Waves

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Sources of Traveling Waves

- Lightning
- Switching Lines or Cables
- Faults
Traveling Wave

Wavetail

Wavefront

Wave velocity and direction
Traveling Waves:
Functions of Time or Distance

- Time, $t$: $f(t \pm \frac{x}{v})$

  or

- Distance, $x$: $f(vt \pm x)$
Time Function

Siemens Industry, Inc., Siemens Power Technologies International

distortionless line

$X_1$
Oscilloscope 1

$X_2$
Oscilloscope 2

$X_3$
Oscilloscope 3

$t_1$ $t_2 > t_1$ $t_3 > t_2$
Wave Velocity

- Waves travel at a constant velocity, $v$:

  \[
  \frac{dx}{dt} = +v \quad \text{Forward wave}
  \]

  \[
  \frac{dx}{dt} = -v \quad \text{Backward wave}
  \]
The travel time $\tau_{12}$ for a wave to move from $x_1$ to $x_2$ is

$$t_2 - t_1 = \tau_{12} = \frac{x_2 - x_1}{v}$$
Travel Time Problem
Waves travel forward & backward

Forward & backward is relative to the choice for positive x
Waves travel forward & backward

At any point in time, or at any position, the total function is the sum of forward & backward waves:

\[ f = f_F + f_B \]

\[ f(t, x) = f_F(t-x/v) \ u(t-x/v) + f_B(t+x/v) \ u(t+x/v) \]
Superposition of Waves

t_1

t_2

t_3
The electric (E) and magnetic (H) field vectors are orthogonal to each other and orthogonal to the direction of propagation.

\[ \mathbf{S} = \mathbf{E} \times \mathbf{H} \]

Power (S) propagates in the direction of the Poynting vector.
Properties of Free Space

- permeability:  \( \mu = \mu_r \times \mu_o \)
  - permeability of free space:
    \[ \mu_o = 4\pi \times 10^{-7} \text{ Henry/meter} \]

- permittivity:  \( \varepsilon = \varepsilon_r \times \varepsilon_o \)
  - permittivity of free space:
    \[ \varepsilon_o = 10^{-9} / 36\pi \text{ Farad/meter} = 8.85 \times 10^{-12} \text{ F/m} \]
TEM Velocity of Propagation

TEM velocity of propagation:

- velocity of propagation in free space:
  
  \[ v_o = 3 \times 10^8 \text{ m/s} = 300 \text{ m/\mu s} \]
  
  \[ v_o = 3 \times 10^5 \text{ km/s} \]
  
  \[ v_o = 186,450 \text{ miles/s} \]
  
  \[ v_o = 984 \text{ feet/\mu s} \]
TEM Intrinsic Impedance

- TEM intrinsic impedance:

\[ \eta = \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} \]

- Intrinsic impedance of free space: \( \eta_o = 377 \ \Omega \)
Dielectric Properties

\[ \eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{377}{300} \]

\[ v = \frac{v_0}{\sqrt{\varepsilon_r}} = \frac{300}{\sqrt{\varepsilon_r}} \]

<table>
<thead>
<tr>
<th>Material</th>
<th>( \varepsilon_r ) relative permittivity</th>
<th>Impedance velocity (ohms)</th>
<th>Propagation velocity (m/µs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vacuum/gases</td>
<td>1.0</td>
<td>377</td>
<td>300</td>
</tr>
<tr>
<td>Oil/paper</td>
<td>3.3 to 3.9</td>
<td>191 to 207</td>
<td>152 to 165</td>
</tr>
<tr>
<td>XLPE</td>
<td>2.3 to 3.0</td>
<td>218 to 249</td>
<td>173 to 198</td>
</tr>
<tr>
<td>EPR</td>
<td>2.8 to 4.0</td>
<td>189 to 225</td>
<td>150 to 179</td>
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<td>PVC</td>
<td>&gt;8</td>
<td>133</td>
<td>106</td>
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<tr>
<td>Polyethylene</td>
<td>2.3</td>
<td>249</td>
<td>198</td>
</tr>
</tbody>
</table>
Voltage & Current Traveling Waves

- **Forward Traveling Wave**
  \[ V = +I \cdot Z_s \]

- **Backward Traveling Wave**
  \[ V = -I \cdot Z_s \]
The ammeter connection shows +I for positive forward current.

The ammeter connection shows -I for positive backward current.
General surge impedance formula:

\[ Z_s = \eta G_f \]

\[ Z_s = \sqrt{\frac{\mu}{\varepsilon}} G_f \]

\[ L = \mu G_f \]

\[ C = \frac{\varepsilon}{G_f} \]

\{high frequencies only\}

\( \eta \) is the intrinsic impedance of the dielectric material

\( G_f \) is the geometric factor
Surge Impedance Problem
Single Phase Geometric Factors

- Coaxial cable:

- One bundle over ground:

  \{high frequencies only\}
Surge Impedance Matrix

Traveling wave relationship:

\[
\begin{bmatrix}
Z_{s11} & Z_{s12} & Z_{s13} \\
Z_{s21} & Z_{s22} & Z_{s23} \\
Z_{s31} & Z_{s32} & Z_{s33}
\end{bmatrix} = \eta \begin{bmatrix}
G_{f1} \\
G_{f2} \\
G_{f3}
\end{bmatrix}
\]

\[
\begin{bmatrix}
V_f \\
I_f
\end{bmatrix} = \begin{bmatrix}
Z_s 
\end{bmatrix} \begin{bmatrix}
I_f
\end{bmatrix}
\]
G\textsubscript{f} Matrix Construction

- Self term:

\[ G_{f11} = \frac{1}{2\pi \ln \left( \frac{D_{1'1}}{r'} \right) } \]

- Mutual term:

\[ G_{f12} = \frac{1}{2\pi} \ln \left( \frac{D_{1'2}}{D_{12}} \right) \]

where:

\[ D_{1'1} = 2h_1 \]

\[ D_{1'2} = \left( (h_1 + h_2)^2 + (x_1 - x_2)^2 \right)^{1/2} \]

\[ D_{12} = \left( (h_1 - h_2)^2 + (x_1 - x_2)^2 \right)^{1/2} \]

average conductor height = conductor height at the tower - \frac{2}{3} \text{sag}
Coupling Factor

\[ \text{Coupling Factor} = \frac{V_2}{V_1} = \frac{Z_{12}}{Z_{11}} \]
Equivalents Looking into Infinite Lines

- Single phase

- Two phase

- Single phase with an initial voltage
Traveling Waves Equivalents

Norton

Thevenin

\[ 2I \quad Z_S \]

\[ 2V \quad Z_S \]
Discontinuities in Surge Impedance

Line & cable junction
Junction of multiple lines
Open end
Short circuit
Series reactor or capacitor
Shunt reactor or capacitor
Different line geometry
Transformers and surge arresters
Motors and generators
Traveling Wave Equivalent
at a Change in Surge Impedance

\[ V_T \]

\[ Z_S \]

\[ Z_T \]

\[ 2V \]
Reflection Coefficients

Voltage reflection coefficient:

\[ \Gamma_v = \frac{Z_t - Z_s}{Z_t + Z_s} \]

Current reflection coefficient:

\[ \Gamma_i = -\Gamma_v = \frac{Z_s - Z_t}{Z_t + Z_s} \]
Reflection Coefficient Problem

<table>
<thead>
<tr>
<th>$Z_t$</th>
<th>$\Gamma_v$</th>
<th>$\Gamma_i$</th>
</tr>
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<tbody>
<tr>
<td>open</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2Z_s$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_s$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.5Z_s$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>short</td>
<td></td>
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</tbody>
</table>
Reflected and Refracted Waves

Change in surge impedance

Incident ⇒

⇒ reflected

Reflected ⇒
Traveling Voltage Waves at Open End

incident (forward) \implies reflected (backward)

incident (forward) \implies
**Traveling Wave at the Junction of 2 Lines**

\[ \text{The reflected wave has a magnitude of } V/3 \]

\[ \Rightarrow \text{The refracted wave continuing to the right has a magnitude of } 2V/3 \]

\[ \Rightarrow \text{The initial wave from the left has a magnitude of } V \]

- Each line has the same surge impedance
Tab 4 - EMTP Traveling Wave Results

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EMTP Simulations

- The waveforms on the following pages are the result of EMTP (Electro-Magnetic Transients Program) simulations of travelling waves on distributed parameter lines.
- For some of the simulations, the line is lossless (no resistance)
- Series resistance is included in the simulations with line loss.
Energize a Line With a DC Source

The line has no losses
Fault at the End of a Line with a DC Source

The line has no losses
Fault at the End of a Line with a DC Source
(continued)

The line has losses
Fault at the End of a Line with an AC Source

The line has losses
Energize a Line with an AC Source

The line has losses

The line has losses
Energize a Reactor Terminated Line with a DC Source

The line has no losses
Energize a Capacitor Terminated Line with a DC Source

The line has no losses

The diagram shows the voltage $V_m$ across the capacitor and the voltage $V_e$ on the line as a function of time. The line is energized by a DC source at time 0, and the voltage waves propagate along the line. The travel times of the line are indicated on the x-axis in seconds (s). The diagram illustrates the transient behavior of the system when a capacitor is connected to a line energized by a DC source.
Effects of Lightning on Utility Systems

- Line Insulation Flashovers
  - usually temporary faults
- Equipment failures
  - Transformers
  - Underground cables
  - Surge arresters
  - Nuisance fuse operations
  - Conductor burnout
  - Insulator punctures
  - Pole splintering
  - Various other problems

Leads to:
- Outages
- Voltage dips
- Customer complaints
- O&M costs
Electrification of the Thundercloud

- Cumulus clouds
- Formation of ice crystals above freezing level
- Charge separation

Diagram showing:
- Positive Charge at 30,000-40,000'
- Negative Charge at 10,000-25,000'
- Freezing Point
- Ground
- Updraft of warm moist air
1. Initial streamer breakdown

2. Stepped leader propagation

3. Upward connecting leader rising up to meet the downward leader

4. The high current return stroke propagates back up the channel the completely ionized channel creating a bright flash of light
Types of Lightning Flashes

- Cloud to Cloud
- Cloud to Ground
- Ground to Cloud
- Ground

Intracloud
Polarity of Lightning Stroke Currents

- **Negative currents**
  - >90% cloud-to-ground flashes
  - one or more strokes

- **Positive currents**
  - <10% of flashes
  - typically just one stroke
  - much higher peak current
  - longer wavetails
  - 10 to 100 times as much energy
Rocket-Triggered Lightning

Rocket

Wire
Rocket Triggered Lightning —
Pre-Launch Conditions

Active Storm Cloud

Electric Field

-7 kV per meter

Ground
Principle of Rocket Triggered Lightning

Active Storm Cloud

Enhanced Electric Field at Tip of Rocket

Ground
Rocket Triggers Lightning Strike

Direction of positive current flow
Triggered Lightning Flash to Test Stand

Lightning attaches to wire
Lightning Flash Measurements

Tall towers have also been used.
Waveshape Parameters

- Peak
- Virtual Front Time = \( (t_{90\%} - t_{30\%}) / 0.6 \)
- Rate of rise = Peak / Virtual Front Time
- Maximum rate of rise
- Time to half value

![Waveshape Parameters Diagram](image-url)
Waveshape Parameters – Example
Rocket Triggered Lightning Stroke Currents
Example of Lightning Surge Current

(Data recorded in EPRI RP2542-1)
Each Lightning Flash Can Have Multiple Strokes

- Stroke 1
- Stroke 2
- Stroke 3
- Stroke 4
- Continuing Current

Current (kA)

- 0
- 10
- 100
- 1000

Time

Up to 1 second

Number of Strokes in Flash

- 1
- 2
- 3
- 4
- 5
- 6
- 7
- 8
- 9
- >10

Percentage of flashes containing indicated number of strokes

Percentage of flashes containing at least as many strokes as indicated
## Lightning Waveshape Characteristics

<table>
<thead>
<tr>
<th>Parameter</th>
<th>First Stroke</th>
<th>Subsequent Stroke</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>% Exceeding the Value</td>
<td>% Exceeding the Value</td>
</tr>
<tr>
<td></td>
<td>95%</td>
<td>50%</td>
</tr>
<tr>
<td>Peak Current, kA</td>
<td>14</td>
<td>31</td>
</tr>
<tr>
<td>Rate of Rise, kA/us</td>
<td>9</td>
<td>24</td>
</tr>
<tr>
<td>Virtual Front Time, us</td>
<td>0.5</td>
<td>1.3</td>
</tr>
<tr>
<td>Time to Half Value, us</td>
<td>30</td>
<td>77</td>
</tr>
</tbody>
</table>

Note: Data shown is for negative downward strokes only
Peak Current Probability Distribution

\[ P = \frac{1}{1 + \left( \frac{I}{31} \right)^{2.6}} \]
Wavetail Features

Region of Rapid Decay [initial decay]
Typical durations 10-300 microseconds.
Note: time to “half value” measurements in this region

Region of Slowed Decay [intermediate decay]
(starts at 10-30% of crest magnitude and can last several milliseconds)

Continuing Current
(typically 10-200 amperes 10-30 milliseconds)
Long Duration Currents

A long duration current recorded in EPRI RP3326 (1993)

An extremely long duration waveshape (1994 triggered lightning data). Courtesy Hydro Quebec/NYSEG
Summary of Main Lightning Parameters

- **Negative downward flashes**
  - one or more strokes
  - Peak current range: 2 kA to more than 200 kA.
  - Majority of cloud-to-ground flashes

- **Positive flashes can have**
  - typically just one stroke
  - much higher peak current
  - longer wavetails
  - 10 to 100 times as much energy
US Lightning Flash Density

1996-2000 Flash Density Map

Global Atmospherics, Inc.
Fault Analysis and Lightning Location System

10 kilometer grid

Jan 1, 1996 00:00:00 GMT
To Dec 31, 2000 23:59:59 GMT
Problem: Number of Strikes to a Substation

- Data:
  - Rectangular area
  - E-W is 100 meters
  - N-S is 50 meters
  - Ground Flash Density is 2 Flashes/km²/year

- Find $N_S = \underline{\quad}$ strikes/year

- Find $1/N_S = \underline{\quad}$ years between strikes
Number of Strikes to an Overhead Line

Eriksson's Formula:

\[ N_S = \frac{N_G \left( b + 28H^{0.6} \right)}{1000} \]

\( N_G \) = Ground Flash Density (Flashes/km²/year)
\( H \) = Height (meters)
\( b \) = Distance between outside conductors (meters)
\( N_S \) = Number of strikes/km/year
Problem:
Number of Strikes to an Overhead Line

Data:
- Height (H) is 10 meters
- Width (b) is 5 meters
- Ground Flash Density \( N_G \) is 2 Flashes/km²/year
- Line is 40 km long

Find \( N_S = \) _______ strikes/year
Lightning Strikes a Power Line

- When lightning hits a conductor at midspan, the current splits in half and initiates traveling waves of very high currents and voltages which moves away from the stricken point.

For example:
- lightning current = 20 kA
- Conductor surge impedance = 400 Ohm
- Voltage = 0.5 \times 20 \times 400 = 4000 \text{ kV}
Problem: Lightning Surge Voltage

Data:
- Lightning stroke is 50 kA peak
- Surge impedance is 400 ohms

Find the peak current and voltage of the travelling wave
Cable Danger Zone

- Strike to tree
- Strike above cable
- 10 meters away from cable
- About 20 meters
Tab 5B – Power Frequency Voltage Stresses

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Power Frequency Voltage Stresses

- Nominal voltage stresses
- Overvoltage stresses
Nominal Voltage Stresses
Peak vs. RMS Value of an Ideal Sine or Cosine Wave

- Steady state analysis assumes continuous perfect sine/cosine waves.
- Ideal sine/cosine waves have just 1 frequency component
- Peak Value = $\sqrt{2}$ times the Root Mean Square (RMS) Value
Ideal Voltages of Three-phase Systems

- Ideal voltages are "balanced" with
  - the same amplitude for each phase
  - angles displaced 120 degrees
  - ABC or Positive sequence rotation
  - As phasors:
    \[ V_A = V \angle 0 \quad V_B = V \angle 240 \quad V_C = V \angle 120 \]

- Three-phase voltages as cosine waves:
The relationship between balanced line-to-neutral and phase-to-phase (line-to-line) voltages:

\[ V_{ab} = V_a - V_b = |V_a|(1-a^2) \]

\[ V_{LN} = \frac{V_{LL}}{\sqrt{3}} \]

**EXAMPLE:**

\[ V_{LL} = 13,200 \text{ Volts} \]

\[ V_{LN} = \frac{V_{LL}}{\sqrt{3}} = \frac{13,200}{\sqrt{3}} = 7,621 \text{ Volts} \]
Voltage relationships in a three-phase system

- **Line-to-Line Voltage** $V_{LL}$ measured between two phases

- **Line-to-Neutral Voltage** $V_{LN}$ measured between one phase and neutral (ground)
Per Unit System

- per unit (p.u.) instead of V or kV
- A per unit value is simply the actual value divided by a base, or reference value.

\[ \alpha_{(pu)} = \frac{\alpha_{(unit)}}{\alpha_{base}} = 0.01 \times \alpha_{(\%)} \]

\[ \alpha_{(unit)} = \alpha_{(pu)} \times \alpha_{base\ (unit)} \]

Where \( \alpha \) is one of many variables such as V, I, S, Z, Y etc.

- Current, power, impedance, admittance and many other parameters can also be expressed in per unit.

- The nominal voltage is typically used as the line-to-line base voltage.
Base Voltage

\[ V_{LL} \text{ (kV)} = V \text{ (pu)} \times V_{LL\text{-base}} \text{ (kV)} \]

\[ V_{LN} \text{ (kV)} = V \text{ (pu)} \times V_{LN\text{-base}} \text{ (kV)} \]

\[ V_{LN} \text{ (kV}_{pk}\) = V \text{ (pu)} \times V_{LN\text{-pk-base}} \text{ (kV}_{pk}\) \]

\[ V_{LL\text{-base}} \text{ is usually the nominal line to line voltage in kV}_{RMS} \]

Volts can also be used as the units

\[ V_{LL} = \sqrt{3} V_{LN} \quad V_{\text{peak}} = \sqrt{2} V_{\text{rms}} \quad V_{LN\text{ peak}} = \sqrt{\frac{2}{3}} V_{LL\text{ rms}} \]
Voltage Base Example for a 138 kV system

- 138 kV is the nominal line-to-line voltage
- 138 kV is also the line-to-line voltage base.
- The phase-to-ground voltage base is
  \[ V_{LN-base} = \frac{138}{\sqrt{3}} = 79.67 \text{ kV}_{\text{rms}} \]
- The peak line-to-ground voltage at 1.00 pu is
  \[ V_{LN-peak} = \sqrt{\frac{2}{3}} \times 138 = 112.68 \text{ kV} \]
- If the voltage is at 1.05 pu (105%), then
  - \[ V_{LL} = 1.05 \times 138 = 144.90 \text{ kV}_{\text{rms}} \]
  - \[ V_{LN} = 1.05 \times 79.67 = 83.66 \text{ kV}_{\text{rms}} \]
  - \[ V_{LN-peak} = 1.05 \times 112.68 = 118.31 \text{ kV} \]
**Nominal and Maximum RMS Voltages**

- **Nominal Voltage**
  - the steady state line-to-line rms voltage that names the system

- **Maximum System Operating Voltage**
  - The highest voltage expected under normal operating conditions at any time and at any point of the system.
  - Typically higher than the nominal voltage

- **Equipment Maximum Voltage Rating**
  - The design level for equipment insulation
  - Typically the maximum system operating voltage
### System Voltage Classes

<table>
<thead>
<tr>
<th>Voltage Class</th>
<th>Voltage Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>LV (Low Voltage)</td>
<td>( V_{LL} &lt; 1 \text{ kV} )</td>
</tr>
<tr>
<td>MV (Medium Voltage)</td>
<td>( 1 \text{ kV} &lt; V_{LL} &lt; 72.5 \text{ kV} )</td>
</tr>
<tr>
<td>HV (High Voltage)</td>
<td>( 72.5 \text{ kV} &lt; V_{LL} &lt; 242 \text{ kV} )</td>
</tr>
<tr>
<td>EHV (Extra High Voltage)</td>
<td>( 242 \text{ kV} &lt; V_{LL} &lt; 1000 \text{ kV} )</td>
</tr>
<tr>
<td>UHV (Ultra High Voltage)</td>
<td>( 1000 \text{ kV} &lt; V_{LL} )</td>
</tr>
</tbody>
</table>
### Class Problem – System Voltages

What are some of the system voltages used by your company?

<table>
<thead>
<tr>
<th>Class</th>
<th>Nominal $V_{LLrms}$</th>
<th>Max operating $V_{LLrms}$</th>
<th>Equipment $V_{LLrms}$</th>
<th>1.00 pu $V_{LNrms}$</th>
<th>1.00 $V_{LNpeak}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MV</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>MV</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>HV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EHV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Overvoltage Stresses
Steady State Overvoltages

- Predominately power frequency
- Last indefinitely

Causes:
- Line-to-ground faults
- Open conductors
- Backfeeding
- Ferranti rise
- Resonance
- Ferroresonance
Distributed Line Parameters

The parameters are distributed along its entire length.

Each $\Delta l$ has

- R series resistance
- L series inductance
- C shunt capacitance
- G shunt conductance
  - very small
  - generally ignored for power flow analysis.
A simple pi equivalent can represent transmission lines less than 200 km where

\[ Z = \ell \cdot (R_1 + jX_1) \text{ Ohms} \]

\[ Y = \ell \cdot jY_1 \text{ Siemens} \]

\( \ell \) is the length of the line

R1, X1 & Y1 are the positive sequence parameters in per unit length

Use a line parameters program (i.e LineProp) for the highest accuracy
Ferranti Rise Overvoltages

- Long lines or cables

- \( \frac{V_E}{V_S} = f \left( X_L, B_C \right) \)

\[
V_E = V_S \left( \frac{-jX_c}{jX_L - jX_c} \right)
\]

\[
V_E = V_S \left( \frac{-\frac{2}{\omega L C}}{\omega L - \frac{2}{\omega L C}} \right)
\]

\[
V_E = V_S \left( \frac{1}{1 - \frac{LC}{2(\omega \ell)^2}} \right)
\]
Resonant Overvoltages

- $V_C/V_S = f \left( X_L, B_C, R \right)$
- Usually a high impedance source & a large capacitance
  - System restoration
  - Backfeeding

\[
V_C = V_S \left( \frac{-jX_c}{R + jX_L - jX_c} \right)
\]

\[
V_C/V_S = \left( \frac{1}{1 - (R + jX_L)/jX_c} \right)
\]
Resonant Overvoltages

for \( R = 0 \)

\[
\frac{V_c}{V_s} = \left( \frac{1}{1 - \frac{X_L}{X_C}} \right)
\]

- Resonant at the power frequency when \( X_L = X_C \)
Backfeeding occurs when a circuit is fed from a lower voltage (non-generator) system.

This circuit could see an overvoltage when bus tie breaker opens.
Ferroresonant Overvoltages

Often a transformer energized through a series capacitance

- Open Circuit Breaker
- Parallel Lines
- Open phase on a distribution feeder

\[ V_T = f( X_L, B_C, R) \]
Overvoltages From Line-to-ground Faults

Function of System Grounding

- Ungrounded
  - source (generator or transformer) has delta winding

- Solidly Grounded
  - source has wye winding with neutral connected to ground

- Impedance Grounded
  - source has wye winding with neutral connected to ground through a reactor or resistor
Voltages On Solidly Grounded Systems

Source transformer winding

Fault to Ground

\[ |V_{CN}| = V_{LN} \]

\[ |V_{BN}| = V_{LN} \]

\[ |V_{AN}| = 0 \]

\[ Z_0 = Z_1 \]
Overvoltages On Ungrounded Systems

\[ |V_{CN}| = V_{LL} \]
\[ |V_{BN}| = V_{LL} \]
\[ V_{AN} = 0 \]
\[ Z_0 = \infty \]

Source transformer winding

Fault to ground
Overvoltages On Impedance Grounded Systems

\[ Z_0 > Z_1 \]

\[ V_{BN} \text{ & } V_{CN} = f(Z_0, Z_1) \]

\[ |V_{AN}| = 0 \]
Temporary Overvoltages (TOV)

- Magnitudes above rated voltage
- Last more than 2 cycles
- A high power frequency component
- Plus the possibility of a higher frequency component
- TOV ratings or capability
  - Curves for surge arresters
  - Limited for most other power delivery equipment
Some Causes of Temporary Overvoltages

- Ferroresonance
- Load Rejection
- Single Line to Ground Faults
- Line Energizing
- Transformer Energizing
- Backfeeding
  - EHV from MV
  - MV from LV
Tab 5C - Switching Transients

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Transient Overvoltages

- Decay with time, usually within one or two cycles

- Often called surges

- The two most common types of transient overvoltages:
  - Switching Surges
  - Lightning Surges
Transient analysis considers many types of time varying functions:

- **Sine & Cosine waves**
  - Power frequency
  - Harmonic frequency
  - Resonant frequency

- **Unit Step function**

- **Exponential functions**

- **Surge functions**
Switching Surges

- Generally contain frequencies from power frequency to tens of kHz
- Caused by the closing or opening of switching equipment (circuit breakers, disconnectors, etc.):
  - Energizing of
    - lines
    - cables
    - transformers
    - reactors
    - buses
  - Line re-energization or high speed reclosing
  - Circuit breaker transient recovery voltages (TRV)
  - Faults
The Ideal Circuit Breaker or Switch Model

When closed

\[ Z = 0 \]

\[ Y = 0 \]

When open

\[ Z = \infty \]

\[ Y = 0 \]
120 V equipment will not conduct until metal-to-metal contact is made.

HV equipment will pre-strike when the dielectric strength of the gap between the contacts is less than the instantaneous applied voltage.
Contacts will pre-strike only near the peak of a power frequency cycle.
Contacts may pre-strike at any point within the power frequency cycle.
Energize Capacitor Bank at Various Angles:

Voltages

(file capclose0.pl4; x-var t) v:CAP 90- v:CAP 45- v:CAP 00- v:SINE

0 4 8 12 16 20 [ms]

-150
-100
-50
0
50
100
150
200

[V]
Energize Capacitor Bank at Various Angles:
Inrush Currents

(file capclose0.pl4; x-var t)  c:CAP 90-     c:CAP 45-     c:CAP 00-

0 4 8 12 16 20 [ms]
-0.700
-0.525
-0.350
-0.175
0.000
0.175
0.350
0.525
0.700 [A]
Three-phase voltages and currents captured by event recorder
- Have only one (or just a few) significant peaks
- Have damped high frequency (100 Hz to 10 kHz) components superimposed on the power frequency voltage
- Waveform from an EMTP simulation of a line energization
Switching Surge Results Are Statistical

- A switching surge magnitude and waveshape depends upon the instant in time that the circuit breaker closes.
- Computer simulations can be repeated with various closing times to obtain a statistical distribution of the overvoltages.
Some equipment uses closing resistors or reactors to reduce transients.

Applications:
- capacitor banks
- EHV lines

Diagram:
- Closes 1st
- Closes 2nd
- Pre-insertion impedance
Analysis of a Voltage Waveform

- Three phase to ground and neutral voltages at the substation bus from simulation of an SLG fault on a network feeder.

Peak of C phase switching surge from fault is 32.8 kV (1.547 pu)

Steady state voltages
Fault starts at peak of A phase
TOV during fault is highest on B Phase
Switching surges from fault clearing
Almost back to steady state voltages

21.21 kV = 1 pu
Saturation Curve for Transformer

Peak Current on LV Winding [kA]
Flux [Webers]

0.47 Wb, 1 kA

air core inductance

0 10 20 30 40 50 60 70 80 90 100
Simulation of a large autotransformer energizing in a weak system

Surge arrester energy >1.4 MJ

V (pu)

i (kA)

I SA
Simulation of a large autotransformer energizing in a weak system
Three-phase set of waveforms of inrush currents from transformer energizing simulation
The Physics of the Interrupting Process

- Process starts when the metallic contacts part
- The last contact point melts & evaporates
- An arc is formed between the contacts
- Arcs contain a high temperature ionized gas (plasma)
- Arcing continues until it is extinguished
Arc Extinguishing

- Arc cooling exceeds arc heating
- Generally occurs at a current zero

![Diagram of current waveform with current zero indicated]
Current Chopping

- no chop

- current chop

normal current zero

A

0 2 4 6 8 10 12

time

0 2 4 6 8 10 12

A
Recovery Voltage

- Current interruption is immediately followed by a voltage across the contacts

- In a very short time, the conducting gas must change into an insulating medium
Resistive Load Switching

- Relatively easy duty

![Diagram showing current, system voltage, and recovery voltage over time](image)

- Current
- System voltage
- Recovery voltage

Time

- Interruption at current zero
Capacitive Current Switching

- Current zero interruption leaves a 1 pu trapped charge on the capacitor
- Recovery voltage reaches 2 pu on 1 phase circuit
Inductive Current Switching

- Relatively small currents
- High frequency transient recovery voltage
- Current chopping can create very high overvoltages

Graph showing time on the x-axis and current/voltage on the y-axis.
CBs must safely handle very high magnitude currents

CBs must open quickly to minimize the impact of the fault on the system
Transient Recovery Voltages (TRV)

- Transient Recovery Voltages (TRV)

- System voltage

- Switch recovery voltage

- Current

- Initial rate of rise

- Crest

- Average rate of rise

- Time
CHAPTER 6
TRANSIENTS IN LUMPED PARAMETER CIRCUITS
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APPENDIX 6B - DYNAMIC SYSTEMS, DIFFERENTIAL EQUATIONS - TRANSIENT AND STEADY STATE SOLUTIONS - OPERATIONAL IMPEDANCE
APPENDIX 6C - LAPLACE TRANSFORMS
APPENDIX 6D - PROBLEMS - LAPLACE TRANSFORMS
APPENDIX 6E - RESPONSE CURVES

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INTRODUCTION

Transients in electrical circuits result from sudden changes in circuit conditions. Faults, lightning strikes and circuit breaker operations all produce transients. Electrical transients usually last only a very short time; an insignificant fraction of the time spent in the steady state. Even so, transients are extremely important because during these periods, the highest voltages and currents occur. Excessive overvoltages or overcurrents can damage equipment or cause circuit tripout.

Electric power engineers should develop an appreciation of transients so as to fully understand the behavior of power systems. However, many engineers do not have any notion of what is happening during transient periods. Sometimes the subject is viewed as bordering on the occult. With patience, transients can be understood, and measures can be taken, through control or design, to make them innocuous.

"Transmission Line Theory II - Traveling Waves and Transients" has been organized into the following five chapters:

Chapter 6 - Transients in Lumped Parameter Circuits is an introduction to power system electrical transients. Lumped parameter circuits are discussed. The bulk of the material is in the appendices.

Chapter 7 - Single Phase Traveling Waves is an introduction to transients in distributed parameter networks. Many important fundamental concepts of traveling waves are introduced here.

Chapter 8 - Multicarrier Traveling Waves continues the discussion of traveling waves as it applies to three phase overhead lines.

Chapter 9 - Lightning Waves on Transmission Lines discusses power systems' most nefarious nemesis.

Chapter 10 - Impulse Overvoltages - Terminals discusses the effect traveling waves have on transformer windings.

SIMPLE TRANSIENTS

The discussion of electrical transients begins with a very simple circuit which has easily understood results. The circuit shown in Figure 6.1 is a dc source, a resistive load and a switch.
This is the circuit of a flashlight, which has a battery, a lamp, and a switch. The solution for the currents and voltages is obvious. With the switch open, the resistor voltage and current is zero. With the switch closed, the resistor voltage is E and the current is E/R. Assuming that the components are ideal, the voltages and currents will instantaneously jump to these values as soon as the switch closes, as shown in Figure 6.2.

Figure 6.1

Figure 6.2
Resistor Voltage for Circuit in Figure 6.1

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The resistor voltage is either $0$ or $E$, depending upon the instant in time. The fundamental concept here is that transient events are functions of time. The starting time of a transient is usually referenced to be an event such as breaker closing or a fault occurrence.

Power systems are primarily ac, so the waveform of sources are sinusoids. Consider the source in Figure 6.1 as sinusoidal. The resistor voltage as a function of time is shown in Figure 6.3.

![Source Voltage Diagram](image)

**Figure 6.3**
Resistor Voltage for Circuit with ac Source

The important concepts shown here are:

1. The peak magnitude is $\sqrt{2}E_{\text{RMS}}$.
2. The value of the resistor voltage at the instant of switching depends upon the instant in the cycle that it occurred.
3. Circuit breakers and switches in power systems normally open at (or near) current zeroes.

The first point above is a common cause of calculation errors. Remember that nominal system voltages are RMS quantities, and refer to line to line potentials on three phase systems. The maximum operating voltage (i.e., 1.05 p.u.) should also be considered in many cases. For example,

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a 345 kV system has a peak voltage (at 1.00 p.u.) of:

\[ V_{\text{peak}} = 345kV \times (\sqrt{2}/\sqrt{3}) = 281.7kV \]

The importance of the second point above will be expounded further in later sections. For now, it will suffice to say that the initial voltage is statistical since circuit breakers can close into a circuit at any point (from 0 to 360°) in a cycle.

The third point is a concern for transients caused by opening actions. A circuit breaker's contacts may begin to part at any point in the current cycle. However, an arc is drawn between the contacts until the current goes through a current zero. Therefore, opening actions tend not to be statistical, since the currents are always interrupted at current zeros.

In wholly resistive circuits, the prediction of transients is simple, but not very interesting since the voltages and currents are no higher than steady state magnitudes. The discussion of wholly resistive circuits ends here because electric power systems have significantly different characteristics.

Electric power networks are dynamic systems. These networks are composed of many elements which can store energy. Energy storage, represented by inductance and capacitance, creates time varying transients as energy is transferred from one component to another.

Dynamic systems are introduced in Appendix 6-B. The discussion of dynamic systems are continued in Appendix 6-C, with the LaPlace Transform analysis.

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REFERENCES


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APPENDIX 2A

BASIC LAPLACE TRANSFORM TABLES
### TRANSMISSION LINE THEORY

#### APPENDIX 6-A

#### BASIC LAPLACE TRANSFORM TABLE

#### Operator Pairs

<table>
<thead>
<tr>
<th>Operator</th>
<th>Expression</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>( f(t) )</td>
<td>( F(s) = \int_0^\infty f(t)e^{-st}dt )</td>
</tr>
<tr>
<td>0-2</td>
<td>( a f(t) )</td>
<td>( aF(s) )</td>
</tr>
<tr>
<td>0-3</td>
<td>( f(t-a) )</td>
<td>( F(s)e^{-as} )</td>
</tr>
<tr>
<td>0-4</td>
<td>( f(t-a) ) ( 0 \leq t &lt; a )</td>
<td>( e^{-as}F(s) )</td>
</tr>
<tr>
<td>0-5</td>
<td>( e^{-at}f(t) )</td>
<td>( F(s+a) )</td>
</tr>
<tr>
<td>0-6</td>
<td>( \frac{df(t)}{dt} )</td>
<td>( sF(s) - f(0) )</td>
</tr>
<tr>
<td>0-7</td>
<td>( \int f(t)dt )</td>
<td>( \frac{1}{s}F(s) + \int_0^s f(\tau)d\tau )</td>
</tr>
</tbody>
</table>

#### Transform Pairs

<table>
<thead>
<tr>
<th>Transform</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-1</td>
<td>impulse</td>
</tr>
<tr>
<td>T-2</td>
<td>( u(t) ) unit step</td>
</tr>
<tr>
<td>T-3</td>
<td>( r(t) = tu(t) ) ramp</td>
</tr>
<tr>
<td>T-4</td>
<td>( e^{-at} )</td>
</tr>
<tr>
<td>T-5</td>
<td>( \frac{1}{a}(he^{-at}) )</td>
</tr>
<tr>
<td>T-6</td>
<td>( \frac{e^{-at}a}{a^2} )</td>
</tr>
</tbody>
</table>
Transform Z-airs (cont'd)

1-7 \( \sin (ut) \)  \[ \frac{\omega}{s^2 + \omega^2} \]

1-8 \( \cos (ut) \)  \[ \frac{s}{s^2 + \omega^2} \]

1-9 \( \frac{1}{u} \) \( (1-\cos ut) \)  \[ \frac{1}{s(s^2 + \omega^2)} \]

1-10 \( e^{-\omega u} \sin ut \)  \[ \frac{\omega}{(s^2 + \omega^2)^2} \]

1-11 \( e^{-\omega u} \cos ut \)  \[ \frac{\omega}{(s^2 + \omega^2)^2} \]
APPENDIX 6B

DYNAMIC SYSTEMS, DIFFERENTIAL EQUATIONS-TRANSIENT AND STEADY STATE SOLUTIONS-OPERATIONAL IMPEDANCE
The study of "control and dynamics" requires the use of certain mathematical tools and techniques which have become an essential part of the technology of control. These tools are all related to methods of solution and analysis of systems described by differential equations. It is not the intent here to go through a detailed theoretical development of the pertinent mathematics that form the basis for the various analysis tools. There are numerous texts that may be referenced for this purpose. The treatment in these appendices will be in the form of a brief review of some basic techniques to supplement and support the material in the main text.

Dynamic Systems

The behavior of dynamic systems is expressed by differential equations relating the systems' variables. In many cases these equations turn out to be or can be approximated by linear differential equations. When this is the case, classical or closed form solutions can be obtained.

For the general case of non-linear differential equations, solutions must be sought through the use of simulation by analog computation methods or by numerical integration techniques carried out on digital computers.

Although any problem can be solved by these simulation methods, the insight that can be derived from linear system analysis is invaluable as a guide to control system design and performance evaluations.

System Differential Equations

Dynamic systems can be thermal, mechanical, electrical or a combination of all these. In order to stay in familiar ground we will illustrate with an electrical example and limit the discussion to linear differential equations.

Consider the circuit in Fig. 8-1.
The differential equation is

$$E = iR + L \frac{di}{dt}$$  \hspace{1cm} B-1

By separating variables, equation B-1 can be put in the form of B-2

$$L \int \frac{di}{i-R} = \int dt$$  \hspace{1cm} B-2

Integration of equation B-2 yields

$$-\frac{1}{L} \ln (i-R) = t + C_1$$  \hspace{1cm} B-3

where $C_1$ is the constant of integration.

Equation B-3 may also be expressed in exponential form as

$$i = i_0 \exp \left(-\frac{R}{L} t\right)$$  \hspace{1cm} B-4

Where $C_2$ is derived from constants of integration which in turn are determined from initial conditions in energy storage elements. The current in inductance $L$ at time $t = 0$ before the switch is closed is $i_0 = 0$.

Substitution of $i = i_0$ at $t = 0$ in equation B-4, yields

$$C_2 = \frac{E}{R}$$  \hspace{1cm} B-5

and equation B-4 can be written as

$$i = \frac{E}{R} \left(1 - e^{-\frac{R}{L} t}\right)$$  \hspace{1cm} B-6
plotted in Fig. B-2 as function of time.

\[ i = \frac{E}{R} \left[ 1 - e^{-\frac{t}{\tau}} \right] \]

\[ \tau = \frac{RL}{R + R_L} \]

FIGURE B-2

This classical solution can be recognized as containing two components:

1. The steady state component
   \[ i_s = \frac{E}{R} \]
   which has the same form as the applied voltage.

2. The transient component
   \[ i_t = \frac{E}{R} e^{-\frac{t}{\tau}} \]
   which decays exponentially to zero.

An alternate method of solution for the current in the circuit of Fig. B-1 is to solve separately for the steady state and transient components as follows:

Let \[ i = i_s + i_t \]

Substituting equation B-5 into equation B-1

\[ E = i_s R + L \frac{di_s}{dt} + i_t R_L + L \frac{di_t}{dt} \]

Since \( E \) is constant \( \frac{di_s}{dt} = 0 \).

By definition also, \( i_s \) and \( \frac{di_s}{dt} = 0 \) in the steady state.

Hence \[ i_t = \frac{E}{R} \]

Substituting equation B-7 into equation B-10 yields the relation from which the transient component may be solved, i.e.:

\[ R_L i_t + L \frac{di_t}{dt} = 0 \]
By definition of the transient component it is exponential in nature, and one can express it as

\[ i_t = I_p e^{pt} \]  \hspace{1cm} (B.13)

Substituting equation B-13 in equation B-12.

\[ (\text{Repl.}) \quad I_p e^{pt} = 0 \]  \hspace{1cm} (B.14)

From equation B-13 the value of \( p \) is determined as

\[ p = \frac{R}{L} \]  \hspace{1cm} (B.15)

which, it can be noted, is independent of the applied voltage \( E \) but merely a function of the circuit parameters.

Substituting equation B-15 in equation B-13 we have

\[ i_t = I_p e^{-\frac{R}{L} t} \]  \hspace{1cm} (B.16)

The value for \( I_p \) is determined from initial conditions, i.e.; the value of \( i \) at \( t = 0 \). The total current \( i = i_L + i_t \)

\[ i = \frac{E}{R} + I_p e^{-\frac{R}{L} t} \]

At \( t = 0 \)

\[ i = \frac{E}{R} + I_p e^{0} = 0 \]

whence \( I_p = -\frac{E}{R} \).

The resultant expression for current is naturally the same as obtained by the classical solution:

\[ i = \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L} t} \]

This example was for a system described by a first order differential equation. For the general system of \( n \)th order the transient component must be chosen in the form \( I_p e^{\lambda t} \). The values of \( \lambda_n \) are evaluated by setting the coefficient of \( I_p e^{\lambda t} \) equal to zero. These principles are covered by other more commonly used methods of differential equation solution such as those which use the Laplace Transform method. We will not pursue the classical method of differential equation solution any further except to introduce the idea of the "characteristic equation" which is basic and which will also be derived with the Laplace Transform method.
Characteristic Equation

The choice of the exponential form for the transient component of the solution of a linear set of differential equations was guided by the results of the classical solution.

This form of solution has the following interesting properties.

If

$$f = e^{pt}$$

then

$$\frac{df}{dt} = pe^{pt}$$

and

$$\frac{d^2f}{dt^2} = p^2e^{pt}$$

Also

$$\int df = \frac{1}{p} e^{pt}$$

Hence in the equation for the transient solution, if \( f \) is substituted by \( le^{pt} \), the terms in the equation

$$\frac{d^n f}{dt^n}$$

are replaced by \( p^n \) and the terms \( \int e^{pt} dt^n \) are replaced by \( 1/p^n \). For instance the differential equation

$$a_0 e^{pt} + a_1 \frac{d}{dt} e^{pt} + \ldots + a_n e^{pt} \int e^{pt} dt^n = f(t)$$

with \( f = le^{pt} \) becomes

$$(a_0 p^n + a_1 p^{n-1} + \ldots + a_n) + \frac{n+1}{p} le^{pt} + \ldots + \frac{a_{n+m}}{p^n} le^{pt} = f(t)$$

The polynomial form of the equation formed by substituting derivatives and integrals by the appropriate \( p \) and \( 1/p \) operators is called the operational form of the equation.

The basic equation which determines the transient modes is independent of the applied forcing function \( f(t) \). It is known as the system characteristic equation and in the example above is

$$(a_0 p^{n-1} + \ldots + a_n + \frac{n+1}{p} \ldots + \frac{a_{n+m}}{p^n}) 0$$

or

$$(a_0 p^{n-1} + \ldots + a_n + \frac{n+1}{p} \ldots + \frac{a_{n+m}}{p^n}) = 0$$
The values of p which satisfy equation B-23 are the roots of the characteristic equation and are the values that appear in the solution $i = I_p e^{pt}$ determining the transient modes of the system.

The characteristic equation of a system and its roots are fundamental to the evaluation of response and stability of dynamic systems.

**Example 1**

Fig. B-3 shows a series RLC network connected to a source $E(t)$ by switch S at $t = 0$.

![RLC Circuit Diagram](image)

**Figure B-3**

The circuit equation for the time after closure of $S$ is

$$Ri + \frac{di}{dt} = \frac{1}{C} \int i \, dt = E(t)$$

B-24

Breaking up the solution into two components (steady state, with same form as $E(t)$ and transient) let us examine the case where $E = \text{constant}$. The steady state solution is found from equation B-24 by noting that $I_S$ has the same form as $E(t)$.

i.e.,

$$\frac{di}{dt} = 0$$

B-25

Substituting equation B-25 in equation B-24.

$$Ri_S + \frac{1}{C} \int i_s \, dt = E$$

B-26

The only way that $i_S$ can be a constant and satisfy equation B-26 is for $i_S = 0$ and $\frac{1}{C} \int i_S \, dt = E$. 
The Transient solution is found by writing the left hand side of equation B-24 in operational form and setting it to zero.

\[ (Lp+R \frac{d}{dp}) i = 0 \]  
\[ p \]

where \( i \) is of the form \( i_e e^{pt} \).

The characteristic equation of equation B-27 is

\[ CRp^2 + (dCp + 1) = 0 \]

which yields the roots

\[ p_1 = - \frac{R}{2C} + \frac{1}{2} \sqrt{(\frac{R}{2C})^2 - \frac{4}{LC}} \]

and

\[ p_2 = - \frac{R}{2C} - \frac{1}{2} \sqrt{(\frac{R}{2C})^2 - \frac{4}{LC}} \]

Depending on whether \((R/L)^2 > 4/(4LC)\) the roots \( p_1 \) and \( p_2 \) will be real or complex.

The expression for the transient current is

\[ i_t = i_1 e^{p_1 t} + i_2 e^{p_2 t} \]

To evaluate \( i_1 \) and \( i_2 \), we note that the system’s initial conditions were \( i = 0 \) and the voltage across the condenser = 0.

\[ i = 0 \]
\[ \frac{1}{l} \int i \, dt = 0 \]

Since the steady state component \( i_2 = 0 \), condition equation B-30.

applies to equation B-29 yields

\[ i_1 = i_2 \]

Also, applying equation B-30 and B-31 to equation B-24 at \( t = 0^+ \)

\[ \frac{d}{dt} \left( i_1 e^{p_1 t} + i_2 e^{p_2 t} \right) \bigg|_{t=0^+} = E \]

\[ B-33 \]
i.e.,

\[ I_1 = \frac{E}{(p_1 - p_2)} \cdot I_2 = \frac{E}{L(p_2 - p_1)} \]

Solving equation B-32 and equation B-33

And the total solution for \( i \) is

\[ I = \frac{E}{L} \cdot \frac{p_1^* \cdot p_2}{p_1^* \cdot p_2} \]

For the case where \( p_1 \) and \( p_2 \) are complex conjugate roots \( (R/L)^2 < 4/LC \),

i.e., where

\[ p_1 = -\alpha + j\beta \]

and

\[ p_2 = -\alpha - j\beta \]

Substituting these values in equation B-35

\[ i = \frac{E}{L} \cdot e^{-\alpha t} \cdot \left( \frac{e^{-\beta t} - e^{-j\beta t}}{2j\beta} \right) \]

which can be expressed as, from the definition of \( \sin \beta t \)

\[ i = \frac{E}{L} \cdot e^{-\alpha t} \cdot \frac{\sin \beta t}{\beta} \]

Figure B-4 shows the nature of the current transient

![Figure B-4](image-url)
Example 2

Take the same example except that let the applied voltage by a sinusoidal function $E = E \cos \omega t$ with the switch again closed at $t = 0$.

Again taking up the steady state solution, equation B-24 becomes

$$R \frac{di}{dt} + L \frac{d}{dt} \left[ \int_{0}^{t} i \, dt \right] = \frac{E}{2} \left( e^{j \omega t} - e^{-j \omega t} \right)$$

where

$$\frac{e^{j \omega t} - e^{-j \omega t}}{2}$$

is the exponential form of $\cos \omega t$.

Since $i_s$ by definition will be of the same form as the applied voltage, we may further divide $i_s$ into components corresponding to the applied voltage components:

$$i_s = i_s e^{j \omega t}$$

where $I_s$ is the complex magnitude of $i_s$ and $I_0$ is the complex magnitude of $i_0$. Considering these components individually, from equation B-39,

$$L \left[ \frac{d}{dt} \left( i_s e^{j \omega t} \right) \right] + \frac{1}{C} \int_{0}^{t} i_s e^{j \omega t} \, dt = \frac{E}{2} e^{j \omega t}$$

or

$$RL \frac{d}{dt} \left[ i_s e^{j \omega t} \right] + \frac{1}{C} \int_{0}^{t} i_s e^{j \omega t} \, dt = \frac{E}{2} e^{j \omega t}$$

Dividing both sides by $e^{j \omega t}$

$$RL \frac{d}{dt} \left[ \frac{i_s}{e^{j \omega t}} \right] + \frac{1}{C} \int_{0}^{t} \frac{i_s}{e^{j \omega t}} \, dt = \frac{E}{2}$$

or

$$1_s = \frac{E}{2 \left( R - j \omega L - \frac{1}{j \omega C} \right)}$$

A similar derivation for $i_0$ yields

$$1_0 = \frac{E}{2 \left( R - j \omega L - \frac{1}{j \omega C} \right)}$$
Equation B-43 can be express as

\[ I_s = \frac{E e^{-j\theta}}{Z} \]  

where

\[ Z = \sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2} \]

\[ \theta = \tan^{-1} \left( \frac{\omega L - \frac{1}{\omega C}}{R} \right) \]

Likewise

\[ I_e = \frac{E e^{-j\theta}}{\omega C} \]

The total steady statecurrent \( i = i_s + i_e \). Using equation B-42 and equation B-43 and substituting equation B-47 and equation B-48

\[ i_s = \frac{E}{Z} \left( \frac{j(\omega t - \theta)}{\omega} \right) - \frac{E}{\omega C} \cos(\omega t - \theta) \]

\[ i_e = \frac{E}{\omega C} \cos(\omega t - \theta) \]

\( Z \) is the impedance of the network and indicates the ratio of voltage to current in the steady state for a sinusoidally varying applied voltage. Equation B-48 is of the same form as the applied voltage \( E \cos \omega t \). Its magnitude is \( E/Z \) and its phase angle with respect to the applied voltage sinusoid is \( \theta \).

The concept of operational impedance \( Z(s) = \frac{1}{s} \) is self-evident from equation B-27.

By substituting \( p = j\omega \) one can derive the impedance to a fixed alternating voltage of frequency \( \omega \) rads/sec.

These concepts are important in the application of "frequency response" techniques which characterize the system in terms of its behaviour as a function of the frequency of the exciting function, \( \omega \).

Although the example was for an electric circuit, yielding the relationship between current and voltage the method is equally applicable to any variables of a system, be they mechanical, electrical or thermal, as long as they are related by linear differential equations.
TRANSMISSION LINE THEORY I
APPENDIX C

LAPLACE TRANSFORMS

The previous sections have reviewed the classical method of solving linear differential equations. We have seen how the transient solution and steady state solution are derived and have developed the concept of operational impedance and impedance to a constant frequency applied excitation function.

These same results can be derived in a greatly simplified fashion through the use of the direct and inverse Laplace transform which uses are approach for both the steady state and transient solution. Laplace transform operational calculus is the corner stone of control system analysis.

Some Basic Theorems of the Laplace Transform

A function of time \( f(t) \) has a Laplace transform \( F(s) \) where

\[
F(s) = \int_{0}^{\infty} f(t) e^{-st} dt
\]

The value of the Laplace transform lies in the fact that a differential equation or expression of the variable "t" transforms into an algebraic equation or expression of the variable "s". This algebraic expression in turn may be operated upon and converted to a form easily recognized in terms of a time function. The process of obtaining the time function from the transform expression is called taking the inverse Laplace transformation. Mathematical operations which in the time domain involve convolution, convert to simple algebraic multiplications in the s domain. A summary of the important theorems governing the use of the Laplace transform are:

1. \[ L \left\{ f(t) \right\} = \int_{0}^{\infty} f(t) e^{-st} dt \]

2. The inverse Laplace transformation \( L^{-1} \) is defined implicitly by the relation

\[ L^{-1} \left\{ \left[ f(t) \right] \right\} = f(t) \quad \text{a} \leq t \]
3. If the functions \( f(t) \), \( f_1(t) \) and \( f_2(t) \) have \( \mathcal{L} \) transforms \( F(s) \), \( F_1(s) \) and \( F_2(s) \) respectively and \( 'a' \) is a constant of a variable which is independent of \( t \) and \( s \), then

\[
\mathcal{L} \left[ a f(t) \right] = a F(s)
\]

and

\[
\mathcal{L} \left[ f_1(t) + f_2(t) \right] = F_1(s) + F_2(s)
\]

Also

\[
\mathcal{L}^{-1} \left[ a F(s) \right] = a f(t) \quad 0 \leq t
\]

and

\[
\mathcal{L}^{-1} \left[ f_1(s) + f_2(s) \right] = f_1(t) + f_2(t) \quad 0 \leq t
\]

4. If a function \( f(t) \) has the \( \mathcal{L} \) transform \( F(s) \), then

\[
\frac{df(t)}{dt} = s F(s) - f(0^+)
\]

where \( f(0^+) \) is the value of \( f(t) \) at \( t = 0^+ \). It is evident then that

\[
\frac{d^2f(t)}{dt^2} = s^2 F(s) - s f(0) - f'(0)
\]

and

\[
\frac{d^n f(t)}{dt^n} = s^n F(s) - \sum_{k=1}^{n} f^{(k-1)}(0) \frac{s^n}{n!}
\]

5. If the function \( f(t) \) has the transform \( F(s) \), its integral \( f^{(-1)}(t) = \int f(t) dt \) has the transform

\[
\mathcal{L} \left[ \int f(t) \ dt \right] = \frac{F(s)}{s} + \frac{f^{(-1)}(0^+)}{s}
\]

Similarly,

\[
\mathcal{L} \left[ f^{(-2)}(t) \right] = \frac{F(s)}{s^2} + \frac{f^{(-1)}(0)}{s^2} + \frac{f^{(-2)}(0)}{s}
\]

and

\[
\mathcal{L} \left[ f^{(-n)}(t) \right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{f^{(k-1)}(0)}{s^{n-k+1}}
\]
6. The Laplace transforms of some common functions are as follows:

\[ f(t) \quad \overset{\mathcal{L}}{\longrightarrow} \quad F(s) \]

- \( f(t) = 0 \) for \( t \leq 0 \)

- \( f(t) = u(t) \)

- \( e^{-at} \)

- \( \frac{1}{b} \sin bt \)

- \( \cos bt \)

- \( \frac{1}{b} e^{-at} \sin bt \)

- \( t \)

- \( \frac{1}{(n-1)!} t^{n-1} \)

- \( t e^{-at} \)

- \( u(t-a) \)

- \( U(t-a) - u(t-a) \)

\[ \text{Unit impulse} \]

\[ u_c(t) = \lim_{a \to 0} \frac{U(t) - u(t-a)}{a} \]

The Laplace Transform

We shall now apply Laplace transform methods to the solution of differential equations.

Take for instance:

\[ A \frac{d^2 y}{dt^2} + B \frac{dy}{dt} + Cy = f(t) \quad \ldots \]
in which A, B, and C are known constants.

The unknown $y(t)$ will be called the response function and the known $f(t)$ will be called the driving function. The initial values of the unknown and its first derivative are $y(0)$ and $y'(0)$.

Applying the $\mathcal{L}$ transformation to both members of equation C-15,

$$\mathcal{L}\left[A \frac{d^2}{dt^2} + B \frac{dy}{dt} + Cy = f(t)\right]$$

C-16

Calling $F(s)$ the $\mathcal{L}$ transform of $f(t)$ and $Y(s)$ the $\mathcal{L}$ transform of $y(t)$, the response transform. Then, using equation C-8 and equation C-9

$$\mathcal{L}\left[y'(s)\right] = sY(s) - y(0)$$

and

$$\mathcal{L}\left[y''(s)\right] = s^2Y(s) - sy(0) - y'(0)$$

This discloses the way in which the initial conditions $y(0)$ and $y'(0)$ are incorporated in the solution during the process of transformation.

Equation C-16 becomes

$$A \mathcal{L}\left[\frac{d^2}{dt^2}\right] + B \mathcal{L}\left[\frac{dy}{dt}\right] + C \mathcal{L}\left[y\right] = \mathcal{L}[f(t)]$$

$$A \left[s^2Y(s) - sy(0) - y'(0)\right] + B \left[sY(s) - y(0)\right] + Cy(s) = f(s)$$

or

$$(A s^2 + Bs + C) Y(s) = F(s) + y(0) (As + B) + y'(0) A$$

C-17

Equation C-17 is called a transform equation. The polynomial coefficient of $Y(s)$—in this case $(As^2 + Bs + C)$—is called the characteristic function since it completely characterizes the physical system described by the differential equation. Note that this is identical with the system characteristic equation derived in Appendix B, except for the variable "s" instead of the operator "p". The equation formed by setting it to zero is called the characteristic equation of the system.

Solving equation C-17 algebraically,

$$Y(s) = \frac{1}{A s^2 + Bs + C} \left[F(s) + y(0) (As + B) + y'(0) A\right]$$

C-18
This algebraic equation has a form which will be found typical of all transform solutions, viz.

Response transform = System function x Excitation function. The system function in this example is the reciprocal of the characteristic function, but in general it will be a fraction of which the characteristic function is the denominator. It incorporates in one function all the essential knowledge regarding the physical system.

The excitation function includes the driving transform and the initial conditions. It contains all the essential specifications of the excitations applied to the system.

When the form of the driving function i(t) is specified, the algebraic form of Y(s) can be determined and

\[ Y(s) = \mathcal{L}^{-1}\left[\frac{Y(s)}{A s^2 + B s + C}\right] \text{ C-19} \]

If Y(s) were an algebraic function of the form of any one of the various transforms listed so far, the inverse could be written immediately by reference to the table. But since Y(s) is a more complicated function than listed, such a direct method of determining the inverse transform fails.

This difficulty may be surmounted by resolving the function into a sum of simpler components whose inverse transforms are readily recognized.

**L^{-1} Transformation**

Consider the general rational algebraic fraction

\[ F(s) = \frac{A(s)}{B(s)} = \frac{a_0 s^{p} + a_1 s^{p-1} + \ldots + a_p}{b_0 s^{q} + b_1 s^{q-1} + \ldots + b_q} \text{ C-20} \]

where \( p < q \)

By solving for the roots of the equation \( B(s) = 0 \), and calling these \( s_1, s_2, \ldots, s_q \), the fraction may be expressed as

\[ F(s) = \frac{A(s)}{B(s)} = \frac{A(s)}{(s - s_1)(s - s_2)(s - s_3)\ldots(s - s_q)} \text{ C-21} \]
and the above may in turn be written as a sum of partial fractions, each partial fraction having for its denominator one of the factors of \( B(s) \).

There will be \( n \) of these partial fractions.

i.e.,
\[
\frac{A(s)}{B(s)} = \frac{K_1}{s - s_1} + \frac{K_2}{s - s_2} + \frac{K_3}{s - s_3} + \cdots + \frac{K_n}{s - s_n}
\]

To evaluate the typical coefficient \( K_k \), multiply both members of equation B-22 by \((s - s_k)\) obtaining

\[
\frac{(s - s_k)A(s)}{B(s)} = K_1 \frac{(s - s_k)}{(s - s_1)} + K_2 \frac{(s - s_k)}{(s - s_2)} + \cdots + K_nK_n \frac{(s - s_k)}{(s - s_n)}
\]

In the fraction forming the left member of equation B-23, \((s - s_k)\) is a factor of both numerator and denominator and should be divided out. Then letting \( s = s_k \), this left member becomes a number, and in the right member all terms except \( K_k \) become zero.

i.e.,
\[
K_k = \left[ \frac{(s - s_k)A(s)}{B(s)} \right]_{s = s_k}
\]

But
\[
\frac{(s_k - s_1)}{(s_k - s_2)} \cdots \frac{(s_k - s_{k-1})}{(s_k - s_{k+1})} \cdots \frac{(s_k - s_n)}{(s_k - s_k)} = \frac{d}{ds} \frac{B(s)}{s - s_k}
\]

so equation C-24 can be written

\[
\frac{A(s)}{B(s)} = \sum_{k=1}^{n} \frac{A(s_k)}{B'(s_k)} \frac{1}{s - s_k}
\]

The actual problem of inverse transformation is now a simple one.

\[ L^{-1} \left[ \frac{1}{s - s_k} \right] = e^{s_k t} \]
The above holds for $\frac{A(s)}{B(s)}$, having first order poles only; i.e., the roots of $B(s)$ being
\[s + s_1 \times (s + s_2)^m \times (s + s_3)^n \times \ldots\]
where $n, m, l = 1$
and $s_1 \neq s_2 \neq s_3 \ldots$

Example:

Find the
\[L^{-1}\left[\frac{s_1 s + s_2}{(s + s_1)(s + s_2)(s + s_3)}\right] = K_1 e^{-s_1 t} + K_2 e^{-s_2 t} + K_3 e^{-s_3 t}\]
in which $s_1, s_2, s_3$ are real numbers, all different.

In which
\[K_1 = \begin{bmatrix} a_1 s + b_1 \\ (s + a_1)(s + a_2) \end{bmatrix} = \begin{bmatrix} -a_1 & a_1 + a_0 \\ s & -a_1 \end{bmatrix} = \begin{bmatrix} -a_2 & -a_1 \end{bmatrix}
\]
\[K_2 = \begin{bmatrix} a_2 s + b_2 \\ (s + a_1)(s + a_2) \end{bmatrix} = \begin{bmatrix} -a_2 & a_2 + a_0 \\ s & -a_2 \end{bmatrix} = \begin{bmatrix} -a_3 & -a_2 \end{bmatrix}
\]
\[K_3 = \begin{bmatrix} a_3 s + b_3 \\ (s + a_1)(s + a_2) \end{bmatrix} = \begin{bmatrix} -a_3 & a_3 + a_0 \\ s & -a_3 \end{bmatrix} = \begin{bmatrix} -a_3 & -a_2 \end{bmatrix}
\]

Special case: One pole lies at the Origin.

In $\frac{A(s)}{B(s)}$ of equation C-21 let $s_1 = 0$, then

\[\frac{A(s)}{B(s)} = \frac{A(s)}{s(s - s_2)(s - s_3)(s - s_4) \ldots (s - s_p)} = \frac{A(s)}{s B_1(s)}\]

where $B_1(s) = \frac{B(s)}{s}$.
The form above occurs frequently. It arises, for example, when the excitation function is a constant step and the system function does not have a pole or a zero at $s = 0$.

The final result can be shown to be

$$\mathcal{L}^{-1} \left[ \frac{A(s)}{s} \right] = \frac{A(0)}{s} + \sum_{k=2}^{\infty} \frac{A(s_k)}{s} B^k(s_k) e^{s_k t} \quad C-27$$

**Example:**

Find

$$\mathcal{L}^{-1} \left[ \frac{a_1 + a_0}{s(s + a)^2 + b^2} \right]$$

Here

$$A(s) = (a_1 s + a_0)$$

$$B(s) = \left( (s + a)^2 + b^2 \right)$$

$$B_1(s) = 2(s + a)$$

and $i \theta, \phi = \alpha + \beta i$ and $\beta > 0$

Using equation C-27,

$$\mathcal{L}^{-1} \left[ \frac{s + a_0}{s(s + a)^2 + b^2} \right]$$

$$\frac{A(0)}{B_1(0)} + K_c e^{-(\alpha + \beta i)t} + K_f (\alpha - \beta i) t$$

where

$$K_c = \frac{a_1 s + a_0}{2 \beta (s + a)}$$

$$K_f = \frac{a_1 - \alpha s + \beta \phi}{2 \beta (s + a)}$$

where

$$s_c^2 = a^2 + b^2$$
and

\[ \phi = \tan^{-1} \left( \frac{a_1 b}{a_0 - a_3} \right) - \tan^{-1} \left( \frac{a_2 b}{a_0 - a_3} \right) \]

Similarly

\[ K = \left( \frac{a_1 b + a_0}{a_0} \right) \]

Coefficients \( K_2 \) and \( K_3 \) are conjugate complex numbers.

The final result can be written

\[ \mathcal{L}^{-1} \left[ \frac{a_1 s + a_0}{b (s^2 + \omega^2)} \right] \]

\[ = \frac{a_0}{b} + \frac{1}{b \sigma} \left[ (a_1 - \omega) s + a_1 \omega \right] + \frac{1}{2 \omega} e^{-\omega t} \sin (\omega t + \phi) \]

A convenient rule to remember in obtaining the \( \mathcal{L}^{-1} \) of a function where one pair of roots are \((s + a)^2 + \omega^2\) is as follows:

The time function corresponding to the roots \( (s + a)^2 + \omega^2 \)

\[ \frac{A(s)}{C(s)} \left[ (s + a)^2 + \omega^2 \right] \]

\[ = \frac{1}{\omega} \sin (\omega t + \phi) \]

where

\[ x = \frac{A(s)}{C(s)} \left( s + a \right) \]

and \( x = \) angle of \( A(-a + \beta) \) minus angle of \( C(-a + \beta) \)

Similarly for a function where one pair of roots are \((s^2 + \omega^2)\) the time function component corresponding to these roots in the function
\[
\frac{A(s)}{C(s)} \left( s^2 + j^2 \right) \quad \text{C-32}
\]

can be obtained as
\[
K \sin (\omega t + \phi)
\]
where
\[
K = \left| \frac{A(j\omega)}{C(j\omega)} \right| \quad \text{C-34}
\]
and
\[
\phi = \text{angle of } A(j\omega) \text{ minus angle of } C(j\omega)
\]

**Multiple Order Poles**

Consider the function \( F(s) \) which has poles of higher order. (\( s_1 \) occurs \( n_1 \) times, \( s_2 \) occurs \( n_2 \) times, etc.)

\[
F(s) = \frac{A(s)}{B(s)} = \frac{A(s)}{(s - s_1)^{n_1} (s - s_2)^{n_2} \ldots (s - s_k)^{n_k}} \quad \text{C-36}
\]

The fraction \( \frac{A(s)}{B(s)} \) may be resolved into a sum of partial fractions. For each pole \( s_k \) of multiplicity \( n_k \), there are \( n_k \) partial fractions

\[
\frac{K_{1k}}{(s - s_k)^{n_k}}, \quad \frac{K_{2k}}{(s - s_k)^{n_k-1}}, \quad \ldots \quad \frac{K_{nk}}{(s - s_k)},
\]

in which the \( K \)'s are constants yet to be determined.

Thus the extension of \( \frac{A(s)}{B(s)} \) is

\[
\frac{A(s)}{B(s)} = \frac{K_{11}}{(s - s_1)^{n_1}} + \frac{K_{12}}{(s - s_1)^{n_1-1}} + \ldots + \frac{K_{1k}}{(s - s_1)^{n_k}} + \ldots + \frac{K_{n1}}{(s - s_1)} + \ldots + \frac{K_{n1}}{(s - s_1)^{n_k}}
\]

\[
+ \ldots
\]

To evaluate the \( K_k \) coefficients, first multiply both members of the equation above by \( (s - s_k)^{n_k} \) obtaining
\[
\left( s - s_k \right)^n \frac{A(s)}{B(s)} = k_{k1} + k_{k2}(s-s_k) + k_{k3}(s-s_k)^2 + \ldots + k_{kn}(s-s_k)^{n-1} + (s-s_k) \frac{k_{k1}}{(s-s_k)^n} + \ldots + \frac{k_{kn}}{(s-s_k)^n}
\]

In the left member \((s - s_k)^n\) cancels out with that factor which is also a part of \(B(s)\). Letting \(s = s_k\), this left member becomes a number which should correspond to \(K_{k1}\) of the right hand side since all other terms would be zero.

In order to obtain the other coefficients, we note that by differentiating both sides with respect to \(s\), the following expression results:

\[
\frac{d}{ds} \left( s - s_k \right)^n \frac{A(s)}{B(s)} = k_{k2} + 2k_{k3}(s-s_k) + \ldots + (n-1)k_{kn}(s-s_k)^{n-2} + (s-s_k) \frac{k_{k1}}{(s-s_k)^n} + \ldots + \frac{k_{kn}}{(s-s_k)^n}
\]

Letting \(s = s_k\), we note that \(k_{k2}\) is equal to the number resulting from the evaluation of

\[
\frac{d}{ds} \left( s - s_k \right)^n \frac{A(s)}{B(s)} \bigg|_{s=s_k}
\]

Similarly for the other terms

\[
k_{k3} = \frac{1}{2!} \frac{d^2}{ds^2} \left( s - s_k \right)^n \frac{A(s)}{B(s)} \bigg|_{s=s_k}
\]

and

\[
k_{kj} = \frac{1}{j!} \frac{d^j}{ds^j} \left( s - s_k \right)^n \frac{A(s)}{B(s)} \bigg|_{s=s_k}
\]
Example:

Find

\[ \begin{bmatrix} a_2 t^2 + a_1 s + a_0 \\ (s+a)^2 \end{bmatrix} \]

\[ L \left[ \begin{bmatrix} a_2 t^2 + a_1 s + a_0 \\ (s+a)^2 \end{bmatrix} \right] = L \left[ \begin{bmatrix} K_{11} + K_{12} t + K_{13} e^{-at} + K_{21} t + K_{22} \end{bmatrix} \right] \]

\[ = \left( K_{11} + K_{12} t + K_{13} e^{-at} + K_{21} t + K_{22} \right) e^{-at} \]

\[ = (K_{11} + K_{12} t + K_{13}) e^{-at} + K_{21} t + K_{22} \]

where

\[ K_{11} = \begin{bmatrix} a_2 t^2 + a_1 s + a_0 \\ (s+a)^2 \end{bmatrix} \]

\[ K_{12} = \begin{bmatrix} a_2 t^2 + a_1 s + a_0 \\ (s+a)^2 \end{bmatrix} \]

\[ K_{13} = \frac{1}{2} \begin{bmatrix} a_2 t^2 + a_1 s + a_0 \\ (s+a)^2 \end{bmatrix} \]

\[ K_{21} = \begin{bmatrix} a_2 t^2 + a_1 s + a_0 \\ (s+a)^2 \end{bmatrix} \]

\[ K_{22} = \begin{bmatrix} a_2 t^2 + a_1 s + a_0 \\ (s+a)^2 \end{bmatrix} \]

Let us complete this section by taking the same examples as in Appendix B.
Consider the circuit of Figure C-1.

The differential equation for condition after closing of the switch at $t = 0$ is

$$ E = R \frac{di}{dt} + \frac{Li}{dt} $$

Taking the transform of both sides of equation C-40

$$ E(s) = Ri(s) + Ls(i(s) - I(0)) $$

where $E(s)$ denotes $\mathcal{L}[E(t)]$,
and $i(s)$ denotes $\mathcal{L}[i(t)]$.
and $I(0)$ is initial condition of $i$ at $t = 0$

Since $E$ is a constant its $\mathcal{L}$ transform is $E/s$ (see table of transforms equation C-14). Also for this case $I(0) = 0$

Hence equation C-49 becomes

$$ E = i(t) \left[ R + Ls \right] $$

Solving for

$$ i(t) = \frac{E}{R + Ls} $$

Equation C-61 is the $\mathcal{L}$ transform solution of the current.

To obtain the time domain solution of current we must perform the inverse transform of equation C-61.

i.e.,

$$ i(t) = \mathcal{L}^{-1}[i(s)] = \mathcal{L}^{-1} \left[ \frac{E}{Ls(s + \frac{R}{L})} \right] $$

C-52
Using the rules of partial fraction expansion, (equation C-22 to equation C-24).

\[ t(t) = \mathcal{L}^{-1} \left( \frac{k_1}{s} + \frac{k_2}{s + \frac{R}{L}} \right) \]

\[ C-53 \]

where

\[ K_1 = \left. \frac{E}{R + i L} \right|_{s=0} = \frac{E}{R} \]

\[ K_2 = \left. \frac{E}{s + \frac{R}{L}} \right|_{s=0} = \frac{E}{R} \]

Using the LaPlace transform tables equation C-14 to obtain the inverse of equation C-53.

\[ t(t) = K_1 + K_2 e^{-\frac{R}{L} t} \]

\[ = \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L} t} \]

\[ C-54 \]

which is the familiar form of the exponential rise in current in the inductive circuit of Figure C-1.

Of particular interest is the exponential term \( e^{-\frac{R}{L} t} \) which reveals the decay of the transient component.

The coefficient \( \frac{L}{R} \) has the dimensions of seconds and is known as the "time constant" of the circuit. This time constant is defined as the time in seconds for the transient term to be reduced to \( e^{-1} \approx 0.369 \) of its initial value. Another useful interpretation of the time constant is the time that would be required for the transient to disappear completely if its rate continued at its initial value. (Fig. C-2)
Take now the case treated in Appendix B of the RLC circuit with the sinusoidal excitation voltage

Again the circuit voltage drop equation is

\[ E(t) = L \frac{di}{dt} + \frac{1}{C} \int \! i \, dt = E \cos \omega t \]

Taking the Laplace transform of both sides of equation C-55

\[ \mathcal{L}\{E(t)\} = \mathcal{L}\{L \frac{di}{dt}\} + \frac{1}{C} \mathcal{L}\{i\} = \frac{E}{\omega^2 s^2 + \omega^2} \]

where \( i(0) = \) initial current at \( t = 0 \)

and \( v_{bc} = \) initial voltage across the capacitor

For the case where these initial conditions are zero, equation C-56 can be expressed as

\[ i(s) = \frac{1}{(R \omega L + \frac{1}{C})} \frac{E_s}{(s^2 + \omega^2)} \]

Note that equation C-57 is in the form
Response transform \( I(s) \) [system function \( \frac{1}{(s+\frac{L}{R})(s+\frac{1}{LC})} \)]

Excitation function \( \frac{E_i}{s} \) [Excitation function \( \frac{E_i}{s} \)]

Expressing equation C-57 in terms of poles and zeros

\[
I(s) = \frac{EC s^2}{(1+CE+CE^2)(s^2+w^2)}
\]

\[
= \frac{EC s^2}{LC(s^2+\frac{B}{E} s^2+\frac{B}{E} s s^2)}
\]

C-59

where the system poles are the roots of \((1+CE+CE^2)\) which are the same as the roots of the characteristic equation C-26 \((1+CE+LC^2)\) in Appendix B.

The time expression for \( I(t) \) is obtained by taking the \( s \rightarrow -s \) of equation C-59 using the rules in equations C-29 to C-35 and expressing equation C-59 as

\[
I(t) = \frac{EC s^2}{LC(s^2+\alpha^2)}
\]

where \( \alpha = 2\alpha \) or \( \alpha = \frac{B}{E} \) and \( \alpha^2 = \frac{1}{4} \left( \frac{B}{E} \right)^2 \)

\[
l(t) = k_1 e^{-\alpha t} \sin (\omega t + \phi_1) + k_2 \sin (\omega t + \phi_2)
\]

C-60

where

\[
k_1 = \frac{EC}{LC} \left[ \frac{s^2 - 2\beta s + \beta^2}{s^2 - 2\beta s + \beta^2} \right]
\]

\[
k_2 = \frac{EC}{LC} \left[ \frac{s^2 - 2\beta s + \beta^2}{s^2 - 2\beta s + \beta^2} \right]
\]

See C-29 to C-31.
\[
\theta_1 = \tan^{-1}\left(\frac{-2\omega}{\alpha^2 - \beta^2}\right) - \tan^{-1}\left(\frac{-2\omega}{\alpha^2 - \beta^2 + \omega^2}\right)
\]

\[
K_2 = \frac{\frac{EC}{\omega \lambda}}{\left((s + \omega)^2 + \omega^2\right)}
\]

(See C-32 to C-35).

\[
K_3 = \frac{K_2}{\left((\alpha^2 + \omega^2)^2 + \alpha^2 \omega^2\right)^{1/2}}
\]

\[
\theta_2 = \pi - \tan^{-1}\left(\frac{2\omega}{\alpha^2 + \omega^2}\right)
\]

Note that equation C-60 has the total solution, steady state (second term) and transient (first term) obtained by a straightforward routine use of the direct and inverse Laplace transform.
APPENDIX 6D

PROBLEMS-LAPLACE TRANSFORMS
PROBLEMS - LAPLACE TRANSFORMS

Problem 6.7 - The unit step is a very useful function in many transient problems. Show the characteristics of a unit step and find the Laplace transform of the unit step.

Solution:

The unit step function is defined as a function which is zero for all negative values of the argument and unity for all positive values of the argument. Often time is the argument of a unit step.

\[ u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \]

A sketch of the unit step shows this definition graphically.

![Graph of the unit step function](image)

The Laplace transform of the unit step can be determined directly from the definition shown as the operators pair D-1 in Appendix A:

\[ F(s) = \int_0^\infty u(t) e^{-st} \, dt = \int_0^\infty e^{-st} \, dt \]

\[ = \left. \frac{e^{-st}}{-s} \right|_0^\infty = 0 - \left( \frac{1}{-s} \right) = \frac{1}{s} \]

This corresponds to the transform pair T-2 in the table of Appendix A.

Comment:

The step can have a magnitude different than unity which can be easily seen by a sketch.

The Laplace transform of a step of magnitude "m" can be found using operator pair D-2.

![Graph of the Laplace transform of a step function](image)
Problem 6.6 - Plot and discuss a delayed unit step. Show how this function can be used as a switching function and develop the Laplace transform for the function.

Solution:

A delayed unit step is a step function which begins at some time, b, as shown below.

\[ u(t-b) = \begin{cases} 
0 & t < b \\
1 & t \geq b 
\end{cases} \]

Recall that a unit step is defined as a function which is zero when the argument is less than zero and equal to unity for all positive arguments. Therefore, to form the unit step we must find a function which is negative for \( t < b \) and positive for \( t \geq b \). The function \( u(t-b) \) satisfies this condition. Therefore

It is interesting to consider the effect of multiplying a delayed unit step times a function of time \( f(t) \) which starts before \( t = b \).

Notice here if we consider the new function \( f_1(t) \)

\[ f_1(t) = u(t-b_1) f(t) \]
this new function is just equal to $f(t)$. The function, $f(t)$, is multiplied by zero for times less than $b_2$ and multiplied by 1 for times greater than $b_2$, thus the function, $f(t)$, is not affected by this operation.

If we now consider a new function $f_2(t)$

$$f_2(t) = u(t-b_2) f(t)$$

the result is different. All of the function, $i(t)$, prior to $b_2$ is multiplied by zero and so the function, $f_2(t)$ becomes

From this example we can interpret the step function $u(t)$ or the delayed step function $u(t-b)$ as a switching function which turns the function on as $b$ and suppresses the argument for all earlier times.

The switching function can also be used as a shifting or displacement function if the function, $f(t)$, is multiplied by $u(t-b_2)$ and if the argument of $f(t)$ is changed from $t$ to $(t-b_2)$, thus forming a new function $f_3(t)$

$$f_3(t) = u(t-b_2) f(t-b_2)$$

The new function is shown in the sketch
A check of this plot can be made by plugging in a series of numbers as the argument to \( f_2(t) \).

The Laplace transform of the delayed unit step can be found using the definition in operation pair 0-1.

\[
F(s) = \int_0^\infty u(t-b)e^{-st}dt
\]

Let \( p = t-b \)

then

\( t = p+b \)

\( dt = dp \)

by substitution then

\[
f(s) = \int_b^\infty u(p)e^{-sp}dp
\]

but note that \( u(p) = 0 \) for \( p < 0 \) therefore the lower limit can be set to zero.

\[
e^{-sb}\int_0^\infty u(p)e^{-sp}dp
\]

\[
F(s) = \frac{e^{-sb}}{s}
\]
From this we see that a delayed unit step becomes the transform for a unit step \((1/s)\) multiplied by \(e^{-sb}\). Therefore \(e^{-sb}\) is referred to as a shifting function or delay function. This shifting function can be used to shift any function of time. From this example we have seen that a transform pair can be set up as

$$u(t-b) \rightarrow \frac{e^{-sb}}{s}$$

If we were to perform the same substitution used above when taking the Laplace transform of the function \(f(t) = u(t-b) f(t-b)\) the resultant transform would be

$$u(t-b) f(t-b) \rightarrow e^{-sb}f(s)$$

Comment:

This delay or shifting function is very useful in transient problems for several reasons. First, this function allows the construction of wave shapes on an incremental basis. This technique will be demonstrated in later problems. Also this shifting function can be used as a basis for analytically writing the equation of traveling waves. When the transmission line equations are solved in the Laplace transform domain the solution comes out in the form \(A(s) e^{-bt}\) as shown in the notes. When any function is of this form it can be interpreted as a delayed function of time which can further be interpreted as a traveling wave.

Problem 6-3 - A ramp function is also a useful function for the analysis of transient phenomena. Discuss the ramp function and its characteristics.

Solution:

A ramp function is a linearly rising function of time. This function can be expressed analytically as the product time, \(t\), and a unit step, \(u(t)\). Notice that \(u(t) = 0\) for a negative argument so in the \((-t)\) region the product \(t \times 0 = 0\).

$$r(t) = tu(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

Notice here that \(r(t)\) is directly proportional to \(t\). If we multiply the ramp function by a constant, \(a\), the new function is no longer only proportional to \(t\) but is equal to "\(a \cdot t\)" for \(t > 0\). This, \(a\), coefficient in effect changes the slope of the ramp function as shown in the sketch.
Comments:

From this example we see that multiplying the ramp function, \( r(t) \), by a constant changes the slope of the ramp. Note here that "a" can be any magnitude and be either positive or negative.

Problem 6.4 - Use the ramp function in problem 6.2 to construct the equation of a truncated ramp.

Solution:

A truncated ramp can be sketched as

\[
r(t) = \alpha_1 \cdot r(t) = \alpha_1 \cdot u(t)
\]
The equation of a line coincident with the front of the truncated ramp is shown on the figure.
The equation for the truncated ramp can be written if we find a function which when added to the
original ramp \( r_1(t) \) produces a horizontal line. If we think of the sum of two ramps we see that
if the slopes are equal and opposite the result is a horizontal line

\[
\begin{align*}
  r_1(t) &= a_1 t + u(t) \\
  r_2(t) &= a_2 t + u(t) \\
  r_3(t) &= (a_1 + a_2) t + u(t)
\end{align*}
\]

This is shown analytically as

\[
r_3(t) = r_1(t) + r_2(t) = (a_1 + a_2) t + u(t) = (a_1 - a_2) t + u(t) = 0
\]

If we delay the second ramp by a time, \( t_1 \), we have the delayed \( r_2(t) \) as

\[
r_2(t) = \text{delayed } r_2(t) = u(t-t_1) r_2(t-t_1) = -a_1(t-t_1) u(t-t_1)
\]

Graphically we see that the truncated ramp function will be generated by adding \( r_1(t) \) and \( r_2(t) \)

\[
r_3(t) = \text{truncated ramp} = r_1(t) + r_2(t) = a_1 t + u(t) - a_1(t-t_1) u(t-t_1)
\]

By addition we can write the equation of \( r_3(t) \) over each region of time as

\[
\begin{align*}
r_3(t) &< 0 \\
r_3(t) &> t_1
\end{align*}
\]

\[
\begin{align*}
r_3(t) &= 0 \\
r_3(t) &= a_1 t \\
r_3(t) &= a_1 t + a_1(t-t_1) = a_1
\end{align*}
\]
The above analytical description of the truncated ramp function appears quite complex but this is because of all of the functional nomenclature. But when working a problem the concept presented here simplifies the solution substantially. If we wish to solve the response of a circuit to a truncated ramp it should be clear from the above discussion that the problem need only be solved once for a ramp driving function.

\[ I(s) = \frac{R(s)}{Z_m(s)} \]

Where \( R(s) = \mathcal{L}[r(t)] \), \( Z_m(s) \) = the Laplace transform of the input impedance. Which might have a plot of the form

\[ r(t) \]

\[ i(t) \]
The result for the ramp \( r_1(t) \) is the above function scaled by \( a_1 \) so this function could be scaled directly from the above result. The response to the second ramp, \( r_2(t) \), can also be scaled from the above results by multiplying by \(-a_1 \) and shifting the result in time. The total response can be added graphically to form the final answer as is sketched below.

Problem 6-5 - Write the differential equations and the Laplace transform for simple inductive and capacitive circuit elements.

Solution:

The differential equation describing the inductive element is

\[
\frac{d}{dt} v(t) = \frac{1}{L} \frac{di}{dt}
\]

The Laplace transform is

\[
v(s) = L \left[ i(s) + i(0^+) \right]
\]
\[ V(s) = sL I(s) - L I(0^-) \]

In many circuit problems there is no initial current in the inductance (i.e., \(I(0^-) = 0\)). For these conditions the circuit equation can be written directly in the Laplace transform sense by replacing \(L\) in the circuit by \(sL\) and writing the equations treating \(sL\) as an impedance.

For example the transformed circuit equations for the following circuit are:

\[ V(s) = R_1 I(s) + sL I(s) + R_2 I(s) \]
\[ Y(s) = (R_1 + R_2 + sL) I(s) \]

Correspondingly the equation for a capacitive circuit element can be written and transformed as

\[ i(t) = C \frac{dV}{dt} \]

The Laplace transform is then

\[ I(s) = C \left[ s V(s) - V(0^-) \right] \]

or

\[ V(s) = \frac{s C}{s} - \frac{V(0^-)}{s} \]

Here the trapped charge on the capacitor at \(t = 0\) is the initial voltage condition. By comparison with operator pair 0-7 we see that \(V(0^-)\) is the initial value of the integral.

For the circuit containing capacitors, the Laplace transform of the differential equation can be written directly by using the value of \(\frac{1}{sC}\) for the circuit element and then treating this element as an impedance element as in the inductance equation above.
APPENDIX 6E
RESPONSE CURVES
Figure 6E-1  
Unit-step response curves  
(Source: "Modern Control Engineering" by Katsuhiko Ogata, Prentice-Hall, Inc.)

$$W_n = \frac{1}{\sqrt{LC}}$$

$$\zeta = \frac{\sqrt{L/C}}{2R}$$  
for parallel RLC circuit

$$\zeta = \frac{R}{2\sqrt{L/C}}$$  
for series RLC circuit

$$\zeta = 1$$ is critically damped
Figure 6E-2 Unit-impulse response curves

(Source: "Modern Control Engineering" by Kazushiko Ogata, Prentice-Hall, Inc.)
CHAPTER 7
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CHAPTER 7

FUNDAMENTALS OF TRAVELING WAVES - SINGLE PHASE

1. INTRODUCTION

The fundamentally unique aspect of transmission lines, relative to the more normal lumped constant circuits, results largely from the finiteness of the velocity of light, and hence to the propagation of electric and magnetic energy along the line as a traveling wave. If a current or voltage change occurs at one end of a long transmission line, the other end does not “know” this change took place until the electrical wave travels the length of the line at the velocity of light. Hence, the termination at the receiving end of this transmission line cannot influence the driving voltage at the source until the wave has both traveled from the source to the receiving end, and then through the interaction of the termination and the transmission line, has sent a response back to the source. Therefore, the electrical signals tend to propagate back and forth as traveling waves, usually dissipating the energy in lossy materials.

A detailed examination of the electromagnetic physics of the problem could result in an analysis showing that guided electromagnetic traveling waves will occur on any open wire overhead transmission line. Further analysis of the electromagnetic equations would show that any low-loss system could also be analyzed by first specifying the inductance and capacitance independently, and then using these lumped system constants and forming the chain or ladder network. The differential equation describing this ladder network would predict the traveling wave effect. The second approach will be used in these notes.

Once the differential equations of the transmission line are used to generate a general solution for the traveling waves on the transmission line, boundary conditions must be used to calculate the specific traveling wave solution. In these notes, the boundary condition problem is referred to as the terminal problem. Solution at the terminals will result in reflection and refraction coefficients which are a function of the various terminations. This terminal condition problem will be approached in two ways, one of which is based on the mathematics of the traveling wave equation and the second of which will be based on an equivalent circuit representation of the transmission line problem, thus reducing the problem to a normal circuit problem.
II. THE TRANSMISSION LINE EQUATION

The equivalent circuit used for calculation of the traveling wave phenomena is the same as that used in Chapter 1 to develop the constant frequency transmission line equation solutions. The equivalent circuit and incremental voltage equation for this circuit are repeated below for easy reference.

\[
\begin{align*}
\mathcal{L}(x,t) & \quad \mathcal{R}(x,t) \\
\Delta x & \\
\mathcal{C}(x,t) & \\
\Delta x & \\
\mathcal{G}(x,t) & \\
\Delta x & \\
\mathcal{V}(x,t) & \\
\Delta x & \\
\mathcal{I}(x,t) & \\
\Delta x &
\end{align*}
\]

where

- \( L \) = henry/meter
- \( C \) = farads/meter
- \( R \) = ohms/meter
- \( \Delta = x_2 - x_1 \)

**FIGURE 7-1**

The current and voltage equations for this circuit are then

\[
\begin{align*}
\mathcal{I}(x, t + \Delta t) &= \mathcal{I}(x, t) - \mathcal{R}(x, t) \mathcal{V}(x, t) - \mathcal{C}(x, t) \frac{\Delta \mathcal{V}(x, t)}{\Delta t} \\
\mathcal{V}(x, t + \Delta t) &= \mathcal{V}(x, t) - \mathcal{L}(x, t) \frac{\Delta \mathcal{I}(x, t)}{\Delta t} - \mathcal{R}(x, t) \mathcal{I}(x, t)
\end{align*}
\]

(7-1)

The current and first partial derivative of the current can be expanded by a Taylor's series expansion as

\[
\mathcal{I}(x, t + \Delta t) \approx \mathcal{I}(x, t) + \frac{\partial \mathcal{I}(x, t)}{\partial t} \Delta t + \ldots
\]

(7-2)

Substituting equation (7-2) (neglecting the higher order terms) into the second equation of (7-1) produces

\[
\begin{align*}
\mathcal{V}(x, t + \Delta t) - \mathcal{V}(x, t) &= -\mathcal{L}(x, t) \frac{\partial \mathcal{I}(x, t)}{\partial t} - \mathcal{R}(x, t) \mathcal{I}(x, t) \\
&= -\mathcal{L}(x, t) \Delta t \frac{\partial \mathcal{I}(x, t)}{\partial t} - \mathcal{R}(x, t) \mathcal{I}(x, t)
\end{align*}
\]
The second order $a^2$ terms will disappear in the limit and using the definition of a derivative.

$$\lim_{h \to 0} \frac{v(x + a(x,t) - v(x,t)}{a} = \frac{dv(x,t)}{dx}$$

obtain

$$\frac{dv(x,t)}{dx} = Ri(x,t) + L \frac{di(x,t)}{dt}$$

By a similar process a second, first order partial differential equation can be developed to produce the pair of equations

$$\frac{dv}{dx} = Ri + L \frac{di}{dt}$$

$$\frac{di}{dt} = vi + 2 \frac{dv}{dx}$$

(7-3)

where

$$v = v(x,t)$$

$$i = i(x,t)$$

These two partial differential equations are used to form the "telegraphers equation" or "transmission line" equations shown below:

$$\frac{d^2 v}{dx^2} = R \frac{dv}{dx} + \left[ \frac{RC + LG}{LC} \right] \frac{dv}{dx} + \frac{d^2 v}{dt^2}$$

(7-4)

$$\frac{d^2 i}{dx^2} = RSI + \left[ \frac{RC + LG}{LC} \right] \frac{di}{dx} + \frac{d^2 i}{dt^2}$$

(7-5)

The solution of these differential equations will provide a function describing the behavior of $v$ and $i$ along the transmission line. The fact that both $v$ and $i$ must satisfy the same differential equation does not mean that the current and voltage are the same function of $x$ and $t$ in a practical problem. The difference will result from the boundary conditions.

Equations (7-4) and (7-5) can be solved in a closed form, but the solutions are quite complex and they are seldom used in the analysis of power system problems. A much simpler solution which clearly demonstrates the concepts is obtained for the lossless case. This simple solution will be used in these notes because the evaluation of transients of very short duration, such as impulse conditions, are generally quite adequately represented by the lossless case.
For $R + G = 0$, the differential equations for voltage and current become:

\[
\frac{\partial^2 V}{\partial t^2} = \frac{L}{C} \frac{\partial V}{\partial t} \tag{7-6}
\]

\[
\frac{\partial^2 I}{\partial t^2} = \frac{G}{C} \frac{\partial I}{\partial t} \tag{7-7}
\]

A. Solution to the Transmission Line Equations

Mathematicians would recognize these equations as the differential equations of the traveling wave problem, but this aspect can easily be deduced by taking the Laplace transform of the equations. For the time being, only the voltage solution will be carried along.

\[
\frac{s^2 V(s,x)}{s^2} = s^2LC V(s,x) \tag{7-8}
\]

where

\[
V(s,x) = \text{the Laplace transform of } V(x,t)
\]

This Laplace transform assumes quiescent conditions for voltage at $t = 0$. The initial conditions can be incorporated later when applying boundary conditions. Solution to equation (7-8) can be found by assuming a solution of the form

\[
V(s,x) = M(s) e^{-mx} \tag{7-9}
\]

where $m$ is the parameter to be evaluated from the differential equation. Substituting equation (7-9) back into equation (7-8) results in

\[
m^2 M(s) e^{-mx} = s^2LC M(s) e^{-mx} \tag{7-10}
\]

For the $m$ to be valid for any and all values of $x$ we see that $m^2$ must be equal to the coefficient on the right hand side of the equation, or

\[
m^2 = \frac{s^2}{s^2LC} \tag{7-11}
\]

or

\[
m = \pm \sqrt{LC} \tag{7-11}
\]

This value of $m$ can be substituted back into equation (7-9), which because equation (7-8) was a second order differential equation, requires two arbitrary constants to be evaluated by the boundary conditions. Thus, a solution can be assumed as:

\[
V(s,x) = A(s) e^{-\sqrt{LC} x} + B(s) e^{+\sqrt{LC} x} \tag{7-12}
\]
This solution can be checked by substituting back into equation (7-8). A complete spatial description for the portion which is a function of \( x \) is contained in the exponential, while the time solution or portion which is a function of \( t \) is included both in the arbitrary constants \( A(s) \) and \( B(s) \) as well as the exponential.

The exercises on the use of the Laplace transform demonstrate the expression \( e^{-2s} \) when transformed to the time domain results in a delayed unit step \( u(t-z) \). It is also easily shown that \( v(t-z) \) is a mathematical representation of a traveling wave. In equation (7-12), the same interpretation can be made; that is, both portions of the voltage solution are traveling waves, thus using the relationship:

\[
\mathcal{L}\left[ u(t-z) \right] = e^{-2s} F(s) \tag{7-13}
\]

where

\[
\mathcal{L}\left[ u(t) \right] = F(s) \]

The time solution to equation (7-12) can be written as:

\[
v(t,x) = 8(t-\sqrt{C} x) - 8(t+\sqrt{C} x) \tag{7-14}
\]

This result can be interpreted as two traveling waves, one in the forward or \( +x \) direction and one in the \(-x \) direction. Both waves are traveling with the velocity:

\[
v = \frac{1}{\sqrt{LC}} \tag{7-15}
\]

Note here that we can deduce the direction of travel of the wave by examining the argument of the \( A \) and \( B \) functions in equation (7-14). Note for the \( A \) function, if \( \frac{dx}{dt} \) increases at the same rate as \( t \), the argument of the \( A \) function remains constant, and therefore must be interpreted as a wave moving in the positive \( x \) direction. The \( B \) function can be interpreted in a similar manner. In this case, \( x \) must increase in a negative direction, while \( t \) increases in a positive sense to maintain the argument of the \( B \) function constant, and therefore, the \( B \) function is interpreted as a negative traveling wave, or wave which travels in the negative \( x \) direction.

B. Relationship Between Voltage and Current Solutions

The voltage solution was developed using equation (7-6) and resulted in two arbitrary constants as a function of \( \tau \), which, as equation (7-14) shows, can be two arbitrary functions of time. Because of the similarity of equations (7-6) and (7-7), a corresponding solution can be obtained for the current equation (7-7). The solution can be written directly in both the Laplace transform domain, which corresponds to equation (7-12), and in the time domain, which corresponds to equation (7-14).
\[ I(s,x) = B(s)e^{-s \sqrt{\kappa} x} + E(s)e^{s \sqrt{\kappa} x} \]  \hspace{1cm} (7-16)

\[ I(t,x) = B(t- \sqrt{\kappa} x) u(t- \sqrt{\kappa} x) + E(t- \sqrt{\kappa} x) u(t) \sqrt{\kappa} x \]  \hspace{1cm} (7-17)

Equation (7-16) and equation (7-17) can also be interpreted as positive and negative traveling current wave solutions. Thus, in these two solutions for voltage and current, there are four arbitrary constants relative to \( x \), which are \( A, B, D \) and \( E \). These four constants were developed and incorporated because equations (7-6) and (7-7) were each second order differential equations. But equations (7-6) and (7-7) were developed from the simultaneous solution of two first-order differential equations in \( t \); therefore, only two arbitrary constants can be sustained by this set of solutions. Thus, \( A, B, C \) and \( D \) are not all arbitrary constants, but in fact, are related in some way. Relationship between the current and voltage equations can be found by substituting the solution (equations (7-12) and (7-16)) back into lossless form of equation (7-3), shown below as the Laplace transform of the desired equations.

\[ \frac{dI(s,x)}{dx} = sL I(s,x) \]  \hspace{1cm} (7-18)

\[ \frac{dV(s,x)}{dx} = \frac{1}{sL} V(s,x) \]  \hspace{1cm} (7-19)

Substituting equations (7-12) and (7-16) into equation (7-18) obtain:

\[ s \sqrt{\kappa} A(s) e^{s \sqrt{\kappa} x} - s \sqrt{\kappa} B(s) e^{s \sqrt{\kappa} x} = \]  \hspace{1cm} (7-20)

\[ sL B(s) e^{s \sqrt{\kappa} x} + sL E(s) e^{s \sqrt{\kappa} x} \]

The terms of like exponentials can be collected together as:

\[ -[s \sqrt{\kappa} A(s)] -sL D(s) e^{s \sqrt{\kappa} x} \]  \hspace{1cm} (7-21)

\[ -[s \sqrt{\kappa} B(s)] +sL E(s) e^{s \sqrt{\kappa} x} = 0 \]

If this equation (7-21) is to be true for all values of \( s \) and \( x \), it is apparent that the coefficients of the exponentials must independently be equal to zero, therefore:

\[ \sqrt{\kappa} A(s) = L D(s) \]  \hspace{1cm} (7-22)

or

\[ D(s) = \frac{1}{\sqrt{\kappa}} A(s) \]

and

\[ \sqrt{\kappa} B(s) = -L E(s) \]  \hspace{1cm} (7-23)

or

\[ E(s) = -\frac{1}{\sqrt{\kappa}} B(s) \]
As shown in equations (7-22) and (7-23) the constants D and E can be expressed as functions of A and B and the solutions of equations (7-12) and (7-16) can be written as:

\[ V(s, t) = A(s) e^{st} \sqrt{\frac{L}{C}} x + B(s) e^{-st} \sqrt{\frac{L}{C}} x \]  
\[ I(s, t) = \frac{B(s)}{C} e^{-st} \sqrt{\frac{L}{C}} x - \frac{B(s)}{C} e^{st} \sqrt{\frac{L}{C}} x \]  

In this expression, the term \( Z_c \) is called the transmission line surge impedance, and because \( I \) and \( C \) are a function only of the geometry of the line, \( Z_c \) or the surge impedance is also a function of the geometry of the line.

\[ \text{Surge Impedance} \cdot Z_c = \left( \sqrt{\frac{L}{C}} + \frac{\varepsilon}{\varepsilon^2} \right) \left( \frac{2 \varepsilon n}{\varepsilon + 1} \right) + \frac{1}{2n} \frac{\varepsilon}{\varepsilon^2} \frac{2n}{\varepsilon + 1} \]  

The impedance \( Z_c \) is referred to as the surge impedance because this impedance gives the relationship between a surge voltage and surge current for a voltage and current which are unidirectional on the transmission line. Also in equations (7-24) and (7-25) we see that the current and voltage waves have basically the same shape with the current wave reduced in magnitude by \( 1/2 \). Also, the current wave traveling in a negative direction has a sign opposite to that of the associated voltage wave. A sketch of the voltage and current waves on a transmission line can be used to give a clear interpretation of the meaning of the minus sign on the negative traveling current wave.

![Figure 7-2](image-url)
The negative sign on the current is not mysterious when we realize that the current has a direction associated with its magnitude and mathematically we use the positive x direction as the positive current direction. The voltage on a transmission line is measured as the magnitude from line to ground, while the current is measured as a magnitude from one point to another longitudinally. A positive current flow in the positive direction is one which produces a positive deflection of an ammeter.

FIGURE 7-3

Thus, when the positive voltage wave passes the ammeter in the plus x direction, the current will enter the plus terminal of the ammeter producing a positive deflection. Correspondingly, if a positive voltage wave traveling in the -x direction passes the ammeter, the current wave will enter the minus terminal and thereby produce a negative deflection on the ammeter. In general on practical problems it is not difficult to keep the correct signs on the currents by more or less common sense approach. The above diagram should be helpful in some situations which may tend to be confusing.

III. TRANSMISSION LINE TERMINATIONS AND DISCONTINUITIES

The previous section has demonstrated that any traveling wave on a transmission line can be resolved into a forward and reverse component of voltage and current. This relationship can be written in a reduced form as:
\[ v + v_f = v_p \quad \text{(corresponds to equation (7-14))} \quad \text{(7-27)} \]
\[ i = i_f + i_p \quad \text{(corresponds to equation (7-17))} \quad \text{(7-28)} \]

where
\[ r = \text{forward} \]
\[ r = \text{reverse} \]

At each point along the line the following relationship must also hold true.

\[ v_f = Z_0 i_f \quad \text{(corresponds to Equation (7-22))} \quad \text{(7-29)} \]
\[ v_p = Z_0 i_p \quad \text{(corresponds to Equation (7-23))} \quad \text{(7-30)} \]

These four relationships, equations (7-27), (7-28), (7-29) and (7-30) are true for any wave at any point on the transmission line and therefore, are true at any discontinuity or termination of the transmission line. Thus, a simultaneous solution of the above four transmission line equations and the equations defining the termination will result in a set of boundary conditions which completely define the problem being analyzed.

A. Line Termination - Open End

For the case of an open end transmission line, we can consider a wave arriving as shown by the unit step in Fig. 7-4.

\[ f(t) \]
\[ V_{in} \]

FIGURE 7-4
Thus, if equations (7-27) and (7-28) are valid at every point, they are valid at the terminal, and therefore can be related to the terminal voltage and current as is shown in equations (7-31) and (7-32).

\[ v_f = 2v_r \]  
\[ i_f = 0 = i_r \]  

Using equations (7-29) and (7-30) and equation (7-32), results in the expression where \( v_r \) is a function of the forward or arriving wave.

\[ i_1 + i_p = 0 = \frac{v_f}{Z_c} - \frac{v_r}{Z_c} \]  
or \( v_r = v_f \)

Therefore the terminal voltage \( v_r \) is

\[ v_r = 2v_f \]  
and from equation (7-32),

\[ i_r = -i_f \]

Thus, we see that if a wave \( v_r \) arrives at an open terminal of a transmission line, the terminal voltage is equal to twice the arriving voltage. The reflected voltage is equal to the arriving voltage and the reflected current is equal to minus the arriving current. This is shown schematically in Fig. 7-5.

\[ \text{FIGURE 7-5} \]
Note in Fig. 7-5, that this double-voltage effect propagates back along the transmission line, and for the case of a unit step, eventually would charge the line fully to 2 per unit voltage. Correspondingly, the current arriving with the forward wave is "wiped out" by the negative reflected current, leaving the line with the zero current between the open terminal and the reflected wave.

Since the original voltage wave propagates down the line toward the open end, the energy in this wave is stored in the magnetic and electric fields around the transmission line. One-half the energy is stored in the electric field (1/2 CV^2) and one-half of the energy is stored in the magnetic field (1/2 LI^2). The energy stored in the region after the reflected wave is passed is of a different nature than that of the original forward traveling wave. Because the current is zero, there is no magnetic energy stored on the line, while the double voltage indicates a higher magnitude of electric energy stored on the transmission line.

B. Line Termination - Resistance

The resistive termination is evaluated in a manner very similar to that for the open circuit line. The additional constraint here is that the terminal voltage and current are related by Ohm's law. The circuit used in the evaluation of this condition is shown in Fig. 7-6.
The following five equations completely define the information known at the terminal of the transmission line.

\[ v_T = v_f + v_r \]  
\[ i_T = i_f + i_r \]  
\[ i_T = v_T \]  
\[ v_f = Z_C i_f \]  
\[ v_r = -Z_C i_r \]

Knowns: \( v_T, R, Z_C \)

Unknowns: \( v_f, i_f, v_r, i_r \)

In these five equations, there are five unknowns as noted; therefore, the solution to the set of equations is feasible. The above equations can be manipulated in the following manner.

\[ i_f = i_r = \frac{v_T}{Z_C} - \frac{v_f}{Z_C} = i_T \]

\[ v_f + v_r = i_T R \]

or

\[ v_T + v_f = \frac{R}{Z_C} v_f - \frac{R}{Z_C} v_f \]

Solving for \( v_f \),

\[ v_f \left(1 + \frac{R}{Z_C}\right) = v_f \left(\frac{R}{Z_C} - 1\right) \]

or

\[ v_f = \frac{R - Z_C}{R + Z_C} v_f \]  
\[ (7-42) \]
The above equations for the reflected voltage wave can be interpreted with respect to the physics of the problem. The line surge impedance and terminal impedance are both resistive. The reflected wave will have the same shape as the forward wave, but will be modified in magnitude as determined by the coefficient for \( v_f \) in equation (7.42). This coefficient of \( v_f \) is commonly called the reflection coefficient for the transmission line:

\[
\text{Voltage Reflection Coefficient } = \left( \frac{R - Z_0}{R + Z_0} \right) \quad (7.43)
\]

The above equations could also be solved for the reflected current as a function of the forward current. The current reflection coefficient then becomes:

\[
\text{Current Reflection Coefficient } = \left( \frac{I_f - \frac{R - Z_0}{R + Z_0} I_0}{I_f + \frac{R - Z_0}{R + Z_0} I_0} \right) \quad (7.44)
\]

Note in equation (7.44) the minus sign enters the problem for the same reason as for the minus sign on the negative traveling current wave.

The above reflection coefficients can be interpreted by considering several specific situations. For instance, if the line is terminated in a resistance \( R = Z_0 \), the reflection coefficient is zero and no reflected voltage or current wave occurs. If \( R \) is less than \( Z_0 \), the reflected voltage wave is negative and the reflected current wave is positive. If \( R \) is greater than \( Z_0 \), the reflected voltage wave is positive and the reflected current wave is negative. These equations can be used to evaluate the reflection coefficient for an open or short circuit condition by letting \( R \) go either infinity or zero respectively.

C. Line Termination - Alternate Interpretation

A common interpretation of the reflection coefficient is useful in many problems, but an alternate method of looking at this problem extends the concept to the point that is more useful in many engineering situations. If we use equations (7.37) and (7.42) to calculate the terminal voltage, we can find the terminal voltage as a function of the arriving or forward wave.

\[
V_f = V_T + v_f \cdot V_T = v_f \cdot \left( \frac{R - Z_0}{R + Z_0} \right) V_T
\]

\[
V_f = 2v_f \cdot \left( \frac{R}{R + Z_0} \right) V_T \quad (7.49)
\]
This equation for terminal voltage can be interpreted as the result of a circuit shown in Fig. 7-7.

![Figure 7.7](image)

Thus we see the circuit representation of equation (7-45) is really just the Thevenin's equivalent representation of the circuit. That is, the system impedance, $Z_c$, is that impedance seen looking into the transmission line with the voltage source shorted and the internal voltage, $2V_f$, is equal to the open circuit voltage with no terminal load on the system. Using this Thevenin's equivalent circuit, the terminal load $R$ can be placed on the circuit and the terminal voltage $V_T$ can then be calculated.

This is a particularly useful circuit in many impulse over-voltage type problems. If we realize that an arriving wave from a transmission line can produce a 2 per unit voltage on the open circuit, the terminal voltage under any resistive load can then be evaluated immediately using the voltage divider concept specifically shown in equation (7-45) and shown diagrammatically in Fig. 7-7.

If $V_T$ is calculated using this equivalent circuit, the reflected voltage can be calculated from equation (7-37),

$$V_f = V_T + V_r$$

or

$$V_r = V_T - V_f$$

(7-46)

This method of calculating the reflected voltage is useful on some circuits where the equivalent circuit approach is used to calculate the terminal voltage directly.
D. Travelling Wave Nomenclature

The concept of a reflected voltage is quite clear and useful on a simple two terminal line. Many times on system with more than two terminals it is convenient to introduce the concept of a refracted voltage wave. The refracted voltage wave in the equivalent circuit is just the terminal voltage. This is clear from a simple example at the junction to two lines.

\[ V_1 \]
\[ \text{LINE 1} \quad Z = 300 \]
\[ V_2 \]
\[ \text{LINE 2} \quad Z = 30 \]

**THE REFLECTION COEFFICIENT**
\[ \Gamma = \frac{30 - 300}{30 + 300} = -\frac{270}{330} = -\frac{9}{11} \]

**FIGURE 7-8**
The reflected voltage wave is \(-9/11\) if \( V_1 = 1 \) and the voltage at the terminal, \( V_2 \), is equal to \( 2/11 \).

\[ V_2 = V_1 + V_R = 1.0 - \frac{2}{11} \]

The voltage propagating to the right on line 2 is equal to \( 2/11 \) and is often referred to as the refracted voltage.*

The concept of a refracted voltage wave is quite useful on simple problems. But for discussion purposes on multiterminal system problems confusion arises when various arriving, reflected and refracted voltage waves are all discussed in the same problem. Therefore, in these notes the concept of a refracted voltage will not be used. The discussion will only refer to arriving and reflected voltage waves and will be defined as:

1. Arriving Wave - any voltage wave arriving to a terminal from a transmission line.
2. Reflected Wave - any voltage wave leaving a bus and propagating down a transmission line.

Using this convention, the voltage on line 2 above would be referred to as a reflected voltage wave.

It may appear that the above rules complicate the calculation of the current in the circuit, because there is a sign difference on the arriving and reflected current waves which will influence the answer. That is, we must know which wave is the forward wave to assign the proper sign convention when calculating the current.

*This is more interesting for a wave passing from a 30 ohm line to a 300 ohm line. The transmitted voltage is higher than the incident voltage in this case. Because of the change in impedance, power is conserved as a simple calculation will show so there is no contradiction in an increased voltage.
By the above convention we have defined the arriving and reflected wave at each terminal of each line. To calculate the current with the correct sign it is necessary to know the forward and reverse voltage wave and use these in equation (7-29) and equation (7-30). So at each terminal of a line it is necessary to assign the "forward" = "reverse" labels on the appropriate "arriving" = "reflected" voltages. The problem becomes one of assigning a direction or "positive x" sense to the line and this then defines the corresponding labels.

![Diagram](image)

**FIGURE 7-3**

In practical problems this 'convention confusion' is not as confusing as it may appear from a general viewpoint. When the problem data are organized in a tabular form the confusion on this point can be eliminated. This problem will be discussed more in Chapter 8.

IV. GENERAL TRANSMISSION LINE TERMINATION

The prior sections considered line terminations both open and resistive and are useful in many practical engineering problems. The same technique described above could be used to evaluate any terminal condition, but the equivalent circuit approach detailed in III-C above is particularly useful in the evaluation of general terminal condition problems.

The above Thévenin's equivalent description used the case of the resistive termination to develop the equivalent circuit approach but did not rely on the fact that the termination was resistive. That is an internal voltage source with a voltage equal to twice the arriving voltage and a source impedance equal to the line surge impedance is a general equivalent circuit. Any load can be placed on this equivalent circuit therefore this equivalent circuit can be used in the evaluation of transient problems even for non-resistive terminations. If we interpret the voltage source in the (s) domain (the Laplace transform of the voltage source) and consider the Laplace transform of the terminal impedance, Z(s), we obtain the equivalent circuit shown in Fig. 7-10.
For this case, a terminal voltage can be calculated as:

\[
v_T(s) = 2V_f(s) \frac{Z(c) - 2V_f(s)}{Z(c) + 2V_f(s)}
\]

(7-47)

where the reflection coefficient can be found from

\[
V_r = \frac{V_T - V_f}{V_T + V_f} \text{ or } V_r(s) = \frac{V_T(s) - V_f(s)}{V_T(s) + V_f(s)}
\]

or

\[
V_r(s) = 2V_f(s) \frac{Z(c)}{Z(c) + 2V_f(s)} \text{ or } V_r(s) = \frac{I(s) - Z(c)}{Z(c) + 2V_f(s)} V_f(s)
\]

(7-48)

Therefore, in the transform sense, the reflected wave can be calculated. In many circuit problems, evaluation of this complex expression is more difficult than calculating the terminal voltage directly from the equivalent circuit. Once the terminal voltage is calculated the reflected wave may be calculated numerically using equation (7-46).

For instance, if the transmission line is terminated in a capacitance:

\[
Z(s) = -\frac{1}{SC}
\]

and the terminal voltage is found as

\[
V_f(s) = 2V_f(s) \frac{\frac{1}{SC}}{Z(c) + \frac{1}{SC}}
\]
or

\[ Y_f(s) = 2V_f(s) \frac{1}{s + \frac{1}{RC}} \]

Evaluation of this differential equation is simple and can be found in most transform tables.

(See Problem 7-12).

This concept can be used for a wide variety of problems. Consider the case of the capacitance tapped onto the center of a very long transmission line.

![Figure 7-11](image)

As indicated, the source impedance for the Thevenin's equivalent circuit is the surge impedance of the line on which the surge is arriving. The capacitance is accounted for as a load on line 1 terminals; while line 2 leaving the junction also acts as a resistive load where \( R \) is equal to the surge impedance of line 2. The resultant equivalent circuit is shown in Fig. 7-12.

![Figure 7-12](image)
V. ESTABLISHING INITIAL CONDITIONS

A fundamental traveling wave solution given in equations (7-14) and (7-17) was developed for zero initial conditions. If this system has been energized for a considerable length of time prior to the impulse or transient voltage problem being considered, some steady-state initial conditions will be established on the system, and it is desirable to incorporate these into the concepts already developed. It may not always be possible to establish precise initial conditions for all situations, but an intuitive approach can be used to establish initial conditions for most engineering applications.

For example, consider a transmission line which is open on both ends and on which a trapped charge exists. It may be of interest to study the results of closing a switch on the end of this transmission line. For this case, shown in Figure 7-13a, we use the Thevenin's equivalent circuit approach. The equivalent internal voltage is equal to the trapped charge on the transmission line and the input impedance is a line surge impedance. This internal voltage is not doubled because it is not a forward or arriving wave, but is rather the total Thevenin's equivalent voltage, and therefore, the total internal voltage on the equivalent circuit.

![Diagram of TRAPPED CHARGE and SYSTEM](image)

**FIGURE 7-13a**

![Diagram of EQUIVALENT CIRCUIT and Zc](image)

**FIGURE 7-13b**
If this system were being driven by a 60-cycle voltage source instead of the trapped charge of the above example, the internal voltage would be equal to the internal voltage on the line or the 60-cycle voltage wave which would be measured under open circuit conditions. Any transient generated by a terminal effect of switching or superimposing a voltage on the terminals will produce a calculated terminal voltage using this equivalent circuit from which the reflected wave can be calculated (see equation (7-46)).

The effective Thévenin's equivalent internal voltage developed from initial conditions in this way, be they a trapped charge or a sinusoidal source voltage, will not be modified by any terminal effects and can only be modified by voltage waves arriving down the transmission line as an arriving wave at these terminals. Therefore, the general equivalent circuits, including initial conditions and any modified voltages occurring from transients arriving from the transmission line, are shown in Fig. 7-14.

![Circuit Diagram]

**FIGURE 7-14**

From this circuit $v_I$ and therefore the reflected voltage can be calculated.
PROBLEMS - TRAVELING WAVE DISCUSSION

Problem 7-6 - Sketch the delayed unit step function with the purpose of interpreting the results as a traveling wave phenomena.

Solution:

The delayed unit step voltage can be plotted on the time axis as before, but this time we can consider two different values of $b$, namely $b_1$ and $b_2$.

\[ u(t-\text{b}_1) \]

\[ u(t-\text{b}_2) \]

we can think of $b_1$ as a space location which is a location $b_1$ from some reference point and we can think of $b_2$ as a space location which is a location $b_2$ from the same point. Now if $b_1$ and $b_2$ were on the same line, correlation of observations at these two locations would suggest a traveling wave; i.e., the unit step wave would arrive at $b_1$ prior to arriving at $b_2$. The difference in time $b_2-b_1$ would be the travel time between these points.

A plot can also be made on the $b$ axis by treating $b$ as the variable rather than $t$. In such a plot the space or $b$ function can be sketched for values of time, say $t_1$ and $t_2$.

\[ u(t_1-\text{b}) \]

\[ u(t_2-\text{b}) \]
Note here that the plot appears different because the "b" term in the argument contains a negative sign, thus the function is zero for values of $b$ greater than $t$. Interpreting these plots as traveling waves we see that the wave travels in the plus $b$ direction as $t$ increases.

Problems 7-7 - Evaluate equation (7-14) of the notes and apply the interpretation used in Problem 7-6.

Solution:

Equation (7-14) is

$$v(t,x) = A(t - \sqrt{LC} x)u(t - \sqrt{LC} x) + B(t + \sqrt{LC} x)u(t + \sqrt{LC} x)$$

As discussed in Problem 7-2, a function times a unit step where both have the same argument can be interpreted in the time-space sense by interpreting the unit step only. That is, if an analysis shows that the unit step propagates as a traveling wave, then the function propagates as a traveling wave. Also we note that the arguments of the two terms on the right side of the equal sign above differ only by sign; therefore, we should be able to look at only one term for a complete evaluation. Consider the function

$$f(x,t) = u(t - \frac{x}{c})$$

Note here

$$v = \frac{1}{\sqrt{LC}} \text{ or } v = -\frac{1}{\sqrt{LC}}$$

depending on the appropriate sign of the function (in equation (7-14)). A plot of $f(x,t)$ is shown below on a time diagram.
Correspondingly a space diagram plot can be made

Here we can calculate the velocity of the wave from \( \Delta x = x_2 - x_1 \) and \( \Delta t = t_2 - t_1 = \frac{(x_2 - x_1)}{v} \). Thus the velocity is

\[
velocity = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{x_2 - x_1}{\frac{(x_2 - x_1)}{v}} = v
\]
So we see that the velocity is equal to \( v \) which is equal to \( \tfrac{1}{\sqrt{LC}} \) as stated in the notes.

**Comment:**

A plot of a unit step with the argument \((1-x/v)\) shows that the unit step propagates as a traveling wave at a velocity \( v \) in the plus \( x \) direction. If we consider the argument \((t-x/v)\) we would interpret this as a wave with a negative velocity in the plus \( x \) direction or a positive velocity in the minus \( x \) direction.

There are several arguments of a function which can be interpreted as traveling waves; that is, the arguments include \( x, t \) and \( v \) is various arrangements. The following unit step functions will produce waves in the plus \( x \) direction.

\[
\begin{align*}
    u(t - x/v) \\
    u(vt - x) \\
    u(v - x/t) \\
    u(t/x - 1/v)
\end{align*}
\]

Note in all cases the switching operations or toe of the wave occurs when the argument is equal to zero.

\[
\begin{align*}
    t - x/v &= 0 \\
    vt - x &= 0 \\
    v - x/t &= 0 \\
    t/x - 1/v &= 0
\end{align*}
\]

Note all of these arguments are formulations of the same equation; i.e., the second equation is the first multiplied by \( v \), etc. Thus it is obvious that other formulations can be found such as

\[
\begin{align*}
    1 - \frac{t}{vt} &= 0 \\
    x/v - t &= 0 \\
    \text{etc.}
\end{align*}
\]
PROBLEMS - WAVE SHAPE CONSTRUCTION

Problem 7-8 - Develop the equation of a square wave using the displaced unit step function. Do this by making a time plot of the function at the location $x = 0$.

Solution:

A time plot of the function at $x = 0$ is

The square wave function can be made up as the sum of two unit step functions, one positive starting at $t_1$ and one negative starting at $t_2$ as shown in the sketch. The equation for this first unit step can be found by examining the argument of the unit step in one of the forms discussed in Problem 6-7. For this problem we can use:

$$t - x/v$$

We want the positive function to switch on at $t = t_1$ and at $x = 0$ therefore for this condition the argument must be equal to zero. This is achieved if we bias the time function by $t_1$ and obtain

$$\text{Arg} = t - t_1 - x/v = 0$$

when $t = t_1$, $x = 0$ then $\text{Arg} = 0$ which satisfies the condition. Therefore the positive unit step is

$$u(t - t_1 - x/v)$$

The equation for the negative unit step can then be written as

$$u(t - t_2 - x/v)$$

and the equation for the square wave is

$$SW = u(t - t_1 - x/v) - u(t - t_2 - x/v)$$

Comment:

Notice that a positive time displacement is achieved by substituting $(t - t_1)$ as a new function of $t$. Of course, a negative time displacement can also be implemented by letting $t_1$ be negative.
Problem 7-9 - Use the results of Problem 7-8 and plot the resultant wave at a new location $x_2$.

Solution:

The equations developed in Problem 7-8 give a complete description of the wave for any $x$ and $t$. The plot shown in Problem 7-8 was for $x = 0$. If we examine the argument of the unit step function using now $x = x_2$, we can find the switching time which occurs when the argument is equal to zero.

\[
\text{Arg} \left( t - t_1 - x/v \right) = \text{Arg} \left( t - t_1 - x_2/v \right) = 0
\]

\[
t = t_1 + x_2/v
\]

Thus the switching time is later by the travel time from 0 to $x_2$ which is $x_2/v$. A plot can then be made as:

Comment:

It should be apparent that any function of $x$ and $t$ can be interpreted in the same manner as the unit step function has been interpreted in this series of problems. This is true because all of the evaluation has emphasized the argument of the function rather than the magnitude of the function.

Problem 7-10 - Use two ramp functions to approximate a double exponential function or impulse function.

Solution:

An impulse function (1.6 x 40 microsecond) is described by a double exponential such that the time to crest is 1.5u seconds and the time to one-half value on the tail is 40u seconds. A sketch of such a wave is shown.
PROBLEMS - TRAVELING WAVES

Problem 7-11 - Find the terminal voltage for a step wave of voltage arriving at a resistor terminated line.

Solution:

The equivalent circuit method will be used in this problem. Assume that the step voltage has a magnitude $V$ so we have the following equivalent circuit of the receiving end of the line.

![Equivalent Circuit Diagram]

The terminal voltage $v_T$ then is

$$v_T = 2V \frac{R}{R + Z_c} = V \frac{2R}{R + Z_c}$$

- If $R = Z_c$, then $v_T = V$
- If $R > Z_c$, then $v_T > V$
- If $R < Z_c$, then $v_T < V$

Comment:

It is apparent from this calculation that the surge level on the open end of the line can exceed the magnitude of the arriving voltage.
Giving the equation
\[ \text{Impulse} = 0.666 \times 10^6 M \ t\ u(t) - 0.6795 \times 10^6 \ (t>1.5 \times 10^{-6}) u(t>1.5 \times 10^{-6}) \]

**Connect:**

It is worth noting here that at some point beyond \( t = 1.5 \times 10^{-6} \) sec, the slope of the resultant impulse can be calculated as the difference between \( a_1 \) and \( a_2 \)

\[ \text{Slope} = a_1 - a_2 = (0.666 - 0.6795) \times 10^6 \]

\[ = -0.0129 \times 10^6 \]

This agrees with the sketch. From this slope it is possible to calculate the time from crest to the zero level as \( t \times \) slope. The calculation shows that the zero level is reached 77\( \times \) seconds after crest. If the calculation were to proceed beyond this point it would be necessary to add an additional function to the series to return the slope to zero. A sketch of 3 required functions and the net sum of these functions are shown below.

![Diagram showing impulse response](attachment:image.png)
This function can be approximated by two ramps, the first of which goes through 1.5u sec. and has a magnitude M. This function will be defined at t = 0 and so the function will be only a function of time, t.

\[ r_1(t) = a_1 u(t) \quad t < 1.5 \times 10^{-6} \]

Therefore \( a_1 = \frac{M}{1.5 \times 10^{-6}} = 0.666 \times 10^6 \)

A second ramp, \( r_2(t) \) starting at \( t = 1.5 \times 10^{-6} \) sec. must have a negative slope and the same \( r_1(t) + r_2(t) \) must pass through 0.5 M at \( t = 40 \times 10^{-6} \) sec.

\[ r_2(t) = a_2 (t - 1.5 \times 10^{-6}) u(t - 1.5 \times 10^{-6}) \]

and

Impulse = \( r_1(t) + r_2(t) = a_1 t u(t) + a_2 (t - 1.5 \times 10^{-6}) u(t - 1.5 \times 10^{-6}) \)

at \( t = 40 \times 10^{-6} \) the impulse magnitude is 0.5 M.

\[ 0.5M - a_1 \times 40 \times 10^{-6} + a_2 \times (40 \times 10^{-6} - 1.5 \times 10^{-6}) \]

where \( a_1 = M/1.5 \times 10^{-6} \)

Solving for \( a_2 \)

\[ a_2 = \frac{M}{(38.5 \times 10^{-6}) \cdot (0.5 - 40/1.5)} \]

\[ = \frac{M}{38.5 \times 10^{-6}} \times 26.16 \]

\[ a_2 = -0.6796 \times 10^6 \]
Problem 7-12 - Find the terminal voltage on a resistive terminated line for a ramp function arriving at the line terminal.

Solution:

The equivalent circuit is the same as for Problem 7-11. The driving voltage \( v \) is now a ramp \( V_t \).

\[ r(t) = V_t t \ \text{u}(t) \]

where \( V_t \) is the slope of the ramp.

The form of the solution is the same as in Problem 7-11 and is

\[ v_T = r(t) \frac{28}{R + Z_L} \]

\[ = V_t \frac{28}{R + Z_L} \ t \ \text{u}(t) \]

Here we see the terminal voltage is also a ramp function with the slope a function of \( R \), \( Z_L \) and \( V_t \).

slope = \( V_t \frac{28}{R + Z_L} \)

note

- If \( R > Z_L \) the slope of \( v_T \) = slope of \( r(t) \)
- If \( R = Z_L \) the slope of \( v_T \) = slope of \( r(t) \)
- If \( R < Z_L \) the slope of \( v_T \) < slope of \( r(t) \)
Comment:

Note here that the rate of rise of the terminal voltage is twice the rate of rise of the arriving wave if the line is open at the terminal. From the above tabulation it is apparent that a resistive termination will reduce the rate-of-rise of the terminal voltage. From this we can see that if a line is terminated by another transmission line, the effective terminating resistance is \( R = Z_c \) and the terminal voltage is equal to the arriving voltage.

If the wave enters a station where several lines are joined together the junction voltage will be lower both in magnitude and rate of rise compared to the arriving voltage.
Note here that $v(t) = r(t) \frac{2B}{\lambda + j\omega C} = r(t) \left( \frac{\frac{j\omega}{2}}{\frac{j\omega}{2} + \frac{1}{C}} \right) = \frac{r(t)}{2}$

Here we see that the rate-of-rise and magnitude of the terminal voltage are 0.5 times the arriving voltage. We see that parallel lines at a bus tends to reduce the severity of surges entering a station.

Problem 7.13 - Calculate the terminal voltage for a line with a capacitive load.

Solution:

The equivalent circuit for the system with an arriving voltage $v(t)$ is

\[
\begin{array}{c}
\text{Z}_c \\
\text{C} \\
\text{r} \\
2v(t)
\end{array}
\]

In the transform since the circuit equation is

\[2v(s) = \left( \frac{1}{\text{Z}_c} + \frac{1}{\text{r}} \right) i(s)\]

where $V(s)$ is the Laplace transform of $v$ arriving.

The current is

\[i(s) = \frac{2v(s)}{\text{Z}_c} = v(s) \frac{2}{\text{Z}_c} \frac{s}{s + \frac{1}{\text{Z}_c}}\]

The terminal voltage is then

\[v_t(s) = i(s) \times \frac{1}{\text{Z}_c} = v(s) \frac{2}{\text{Z}_c} \frac{1}{s + \frac{1}{\text{Z}_c}}\]

If the arriving voltage is a unit step

$v(t) = u(t)$ then $V(s) = \frac{1}{s}$
and the terminal voltage is

$$V_T(s) = \frac{2}{X_C C} \frac{1}{s(s + \frac{1}{X_C C})}$$

$$= a \frac{2}{s(s + a)}$$

where $a = \frac{1}{X_C C}$

Using the transform pair T-5 we find

$$V_T(t) = 2a \left[ \frac{1}{a} (1 - e^{-at}) \right] = 2 (1 - e^{-at})$$

Or for an arriving unit step of voltage the terminal voltage is a rising exponential function as shown in the sketch below.

![Sketch of terminal voltage](image)

**Comment:**

Note here that the influence of the capacitive termination is to slope the front of the wave at the line terminal. If the capacitor were removed, the unit step would have resulted in a unit step of double magnitude. The capacitor keeps the terminal voltage from doubling immediately. If the arriving wave magnitude is maintained the voltage will eventually double.
The above result can be used to find the terminal voltage for a ramp function applied. Observe that a ramp function is the integral of a unit step and therefore the terminal voltage response for a ramp will be equal to the integral of the response for a unit step.

\[ V_{T2} = \int V_c(t) \, dt = \int 2(1-e^{-at}) \, dt \]

\[ = \left[ t + \frac{e^{-at}}{a} \right] + K \]

The boundary condition \( V=0 \) at \( t=0 \) can be used to evaluate the constant \( K \)

\[ 0 = 2 \left[ 0 + \frac{e^{0}}{a} \right] + K \]

\[ K = -\frac{2}{a} \]

Therefore the response for a ramp driving function is

\[ V_{T2} = 2 \left[ t - \frac{1}{a} (1 - e^{-at}) \right] \]

A sketch of the response is shown below.

This plot shows that the effect of the capacitor is to reduce the initial rate of rise of the terminal voltage from that which would have existed with no capacitance on the termination.
Chapter 8
MULTIPHASE TRAVELING WAVES

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TRANSMISSION LINE THEORY II
CHAPTER 8
MULTIPHASE TRAVELING WAVES

I. Introduction

The evaluation of transients or surges on power transmission circuits is generally performed on a single phase basis. This is because the calculation of transients is a tedious process and to minimize the effort required, single phase equivalents are used. But with the increased use of digital computers in the calculation of transients the tedious can be reduced. Because of the current interest in the development of multiphase digital computer transient programs some attention to this problem should be given. It is also important to be aware of the assumptions made when studying a multiphase problem on a single phase basis. Some insight to this aspect of this problems will be presented in this chapter.

II. Development of Multiphase Traveling Wave equations

The single phase partial differential equations were developed in Chapter 6 using the distributed L.C equivalent for a line segment $dx$ long and then taking the limit as $dx \to 0$. The single phase transmission line equations are:

$$\frac{\partial^2 V}{\partial x^2} + R \frac{\partial V}{\partial x} + L \frac{\partial I}{\partial t} = 0 \quad (8.01a)$$
$$\frac{\partial I}{\partial x} + G V + C \frac{\partial V}{\partial t} = 0 \quad (8.01b)$$

A similar set of equations can be developed for the multiphase transmission line. This is most easily done in matrix nomenclature and if the steps in Chapter 6 are followed where each term in the equation are matrices, then the following set of simultaneous matrix equations can be developed.

$$\frac{\partial \mathbf{V}}{\partial x} = [\mathbf{R}] \mathbf{I} + [\mathbf{L}] \frac{\partial \mathbf{V}}{\partial t} \quad (8.02a)$$
$$\frac{\partial \mathbf{I}}{\partial x} + [\mathbf{G}] \mathbf{V} + [\mathbf{C}] \frac{\partial \mathbf{V}}{\partial t} = 0 \quad (8.02b)$$

where $[\mathbf{V}]$ is a column vector of conductor voltages
$[\mathbf{I}]$ is a column vector of conductor currents
$[\mathbf{R}]$ is a square matrix of longitudinal resistance
$[\mathbf{L}]$ is a square matrix of longitudinal inductance
$[\mathbf{G}]$ is a square matrix of shunt conductive elements
$[\mathbf{C}]$ is a square matrix of shunt capacitance
In these equations the elements of $[x]$ and $[i]$ are the respective conductor voltages and currents. Of course the order of the elements of the voltage and current must be the same; i.e., if the third element in the voltage column is the voltage for conductor $n$, the third element in the current column must be the current for conductor $n$. The corresponding $[R]$, $[L]$ and $[C]$ matrix elements must correspond to this same matrix order for proper description of the transmission line as discussed in Chapter 3.

This set of simultaneous equations can be formed into the second order differential equation recognized as the traveling wave equation in the single phase sense. The second order voltage differential equation is shown below.

$$\frac{d^2}{dx^2} [V] = [R][G][V] + \left( [R][C] + [L][G] \right) \frac{dV}{dx} + [L][C] \frac{d^2V}{dx^2} \tag{8.03}$$

Here it is possible to take the Laplace transform of equation (8.02) resulting in a second order differential equation in $x$. The equation and the resultant solution in $x$ is shown:

$$\frac{d^2}{dx^2} [V] = [s]^2 [V] \tag{8.04}$$

where $[s]^2 = [s][s] = \text{The Laplace transform of the voltage}$

$$[s]^2 = \left( [R][G] + s \left( [R][C] + [L][G] \right) + [s][L][C] \right)$$

The solution to equation (8.04) is

$$[x] = e^{s}[1] \times [A(s)] + e^{s}[1] \times [B(s)] \tag{8.05}$$

Here $[A(s)]$ and $[B(s)]$ are column vectors which must be evaluated by boundary conditions. While equation (8.05) appears quite simple it is important to remember equation (8.05) is in the Laplace transform space where $[s]$ is a square matrix containing the parameter "s".

There is no simple inverse transform for such an equation and study of this equation shows that there is no simple solution for this equation even in the constant frequency sense; i.e., when $s=0$ is substituted for $s$. This results because the form of the solution above in equation (8.05) includes the term "e" to a matrix exponent. There is no clear straightforward interpretation to this problem but there are mathematical methods of solution available. If the problem is converted to a constant frequency problem the solution in essence requires finding the eigenvectors and eigenvalues of the $[s]$ matrix. *If such a

*This is the procedure followed in constant frequency problems such as carrier current propagation and radio noise calculations. Appendices 3B and 3C treat the eigenvalue solution of $[s]$ and give physical interpretations and practical results. These appendices should be studied in conjunction with this chapter.
procedure is followed for a large number of frequencies the transient solution could be obtained through the use of the Fourier transform and its inverse. This whole problem is quite complex and is being studied by many researchers throughout the world at this time. In these notes no attempt will be made to proceed to a general solution of the multiphase transmission line equation with losses.

But if \([\mathbf{y}] = [\mathbf{G}] \cdot [\mathbf{z}]\) a solution to equation (8.04) can be obtained. Observe that using the matrix concepts from Chapter 2, the product

\[
[\mathbf{L}] \cdot [\mathbf{C}] = k_1 k_2 [\mathbf{I}]
\]

where \([\mathbf{I}]\) is the identity matrix

\[
k_1 = 0.3218 \times 10^{-3} \text{ henry/mile}
\]

\[
k_2 = 0.0895 \times 10^{-6} \text{ farads/mile}
\]

It can be shown that the solution which corresponds to equation (8.06) is

\[
[\mathbf{V}] = e^{-s \sqrt{k_1 k_2}} [\mathbf{A}(s)] + e^{s \sqrt{k_1 k_2}} [\mathbf{B}(s)]
\]

\[
(8.06)
\]

where \(e^{-s \sqrt{k_1 k_2}}\) is a multiplier of each element of the column and defines the propagation and \(e^{s \sqrt{k_1 k_2}}\) is the same for each conductor.

\([\mathbf{A}(s)]\) and \([\mathbf{B}(s)]\) are column vectors of voltage on each conductor which are defined by boundary conditions.

From this solution we can see that the voltage on each conductor propagates undistorted and the problem will, in many respects for the \(n\) conductor system, be similar to the \(n\) single phase problems running in parallel. This should be evident by observing that each of the elements of the \([\mathbf{A}(s)]\) and \([\mathbf{B}(s)]\) matrix are multiplied by \(e^{-s \sqrt{k_1 k_2}}\) which will produce a shifting or transport delay of each of the \(n\) voltage functions of the solution.

A corresponding solution could be obtained for the current differential equations showing a solution of the same form as that in equation (8.06) but in the case of the current equation the constants are different. This is very similar to the single phase case.
Again, as in the single phase problem, it is possible to find the relationship between the constants in equations (8.06) and (8.07) by plugging these solutions into equations (8.02a). Such a procedure can be used to show

\[
[D(s)] = [Z]^{-1} [A(s)] \tag{8.08a}
\]

\[
[E(s)] = -[Z]^{-1} [B(s)] \tag{8.08b}
\]

where

\[
[Z]^{-1} \text{is the inverse of}[Z]
\]

\[
[Z] = \begin{pmatrix}
L_1 & (L_1 C_1)^{-1} \\
C_0 & 1
\end{pmatrix}
\]

\[
120[A]
\]

where the \([A]\) matrix was defined in Chapter 3.

Here the \([A]\) matrix contains the natural algorithms of the distances between the conductors and images as defined in equations (3.04) and (3.11). This expression for the surge impedance matrix is valid for the lossless system but would be somewhat more complex if losses were maintained in the solution.

The Laplace transform of the traveling waves are given in equations (8.06) and (8.07). These voltage and current equations can be interpreted as forward and reverse waves in the same manner as in the single phase case. If the column vectors were written out on a term by term basis it would be evident that each phase or conductor wave would propagate at the speed of light in the case of overhead lines.

For instance consider the first term in equation (8.06) (the forward wave component) for a two conductor line

\[
[V_{1s}(s)] \text{ forward } = e^{-s \sqrt{\frac{V_1}{k_1^2}}} \times [A(s)] \tag{8.09}
\]

Now for a 2 conductor line this equation becomes

\[
\begin{bmatrix}
V_{1s}(s) \\
V_{2s}(s)
\end{bmatrix} = e^{-s \sqrt{\frac{V_1}{k_1^2}}} \begin{bmatrix}
A_1(s) \\
A_2(s)
\end{bmatrix} = \begin{bmatrix}
e^{-s \sqrt{\frac{V_1}{k_1^2}}} \times A_1(s) \\
e^{-s \sqrt{\frac{V_1}{k_1^2}}} \times A_2(s)
\end{bmatrix} \tag{8.10}
\]
This can be written as two simultaneous equations as

\[ V_1(s,x) = e^{-s \sqrt{\frac{1}{k_1^2} x}} A_1(s) \]  
\[ V_2(s,x) = e^{-s \sqrt{\frac{1}{k_2^2} x}} A_2(s) \]  

(8.11)

Here each of these equations can be independently transformed to the time domain as

\[ v_1(t,x) = A_1 \left( t - \frac{x}{v} \right) u \left( t - \frac{x}{v} \right) \]  
\[ v_2(t,x) = A_2 \left( t + \frac{x}{v} \right) u \left( t + \frac{x}{v} \right) \]  

(8.12)

where

\[ v = \frac{1}{\sqrt{k_1 k_2}} \]

Note \( v_1 \) and \( v_2 \) are two forward waves, one on each conductor, which propagate at the same velocity with no distortion or attenuation.

For convenience in these notes a shorthand notation will be used as was done in Chapter 6 (equations (6.27) and (6.28)). Equation (8.12) will be written as

\[ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} L_{v1} \\ L_{v2} \end{bmatrix} \]

(8.13)

Using this nomenclature the results of equations (8.06), (8.07) and (8.08) can be written as

\[ \begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} v_f \end{bmatrix} + \begin{bmatrix} v_r \end{bmatrix} \]  
(8.14a)

\[ \begin{bmatrix} i \end{bmatrix} = \begin{bmatrix} i_f \end{bmatrix} + \begin{bmatrix} i_r \end{bmatrix} \]  
(8.14b)

\[ \begin{bmatrix} v_f \end{bmatrix} = [\mathbb{I}] \begin{bmatrix} i_f \end{bmatrix} \]  
(8.14c)

\[ \begin{bmatrix} v_r \end{bmatrix} = -[\mathbb{I}] \begin{bmatrix} i_r \end{bmatrix} \]  
(8.14d)

where

- \( [v] \) is a column vector of voltages
- \( [i] \) is a column vector of currents
- \( f \) = subscript referring to a forward wave
- \( r \) = subscript referring to a reverse wave
- \( [\mathbb{I}] \) is a square surge impedance matrix

The above equations are just a convenient method of writing out the more complex mathematical equations. It is convenient to think of each element of the above column matrices as functions and in simple problems waves such as impulses will be talked about as forward waves or \( [v_f] \).
III. Multiphase Reflection Coefficient

The reflection coefficient for the multiphase problem follows very much the same procedure used in the single phase case but now the arithmetic must be performed in matrix nomenclature. The simple case of the open and short-circuited line will not be performed here because the results can be deduced directly from the resistance termination results. Suffice it to say that for a terminal with all conductors open the reflection coefficient on each conductor is +1 and for a terminal with all conductors shorted, the reflection coefficient on each conductor is -1.

To find the reflected wave we can schematically show an "n" conductor line and a resistive termination where all elements are now matrices.

A simple example which makes the above diagram somewhat easier to comprehend is shown in Figure 8.02.
The sketch in Figure 8.02 has reduced the problem to a three phase case and includes only a simple resistive termination. In general the terminal impedance can include coupling between phases which would contribute to off diagonal terms in the \([R]\) matrix.

The matrix equations describing the line (equations (8.14)) must be combined with the equations which describe the terminal conditions. Here we will refer to the waves as arriving, "a", wave and reflected, "r", waves.

\[
\begin{align*}
[v_a] &= [R] [i_l] \\
[v_r] &= [v_a] + [v_p] \\
i_p &= [i_p] + [i_p] = [2]^{-1} \left( [v_a] - [v_p] \right)
\end{align*}
\]  

(8.15a)  
(8.15b)  
(8.15c)

Substituting (8.15a) and (8.15c) into (8.15a) obtain

\[
[v_p] = [v_p] - [R] [2]^{-1} \left( [v_a] - [v_p] \right)
\]  

(8.16)

Separating \([v_a]\) and \([v_p]\) obtain

\[
\left( [1] - [R] [2]^{-1} \right) [v_a] = -\left( [1] + [R] [2]^{-1} \right) [v_p]
\]  

(8.17)

Premultiply both sides by \([R]^{-1}\) we have

\[
\left( [R]^{-1} - [2]^{-1} \right) [v_a] = -\left( [1]^{-1} + [2]^{-1} \right) [v_p]
\]  

(8.18)

Solving for \([v_p]\)

\[
[v_p] = \left( [R]^{-1} + [2]^{-1} \right)^{-1} \left( [2]^{-1} - [R]^{-1} \right) [v_a]
\]  

(8.19)

Thus we have a multiphase reflection coefficient which is a function of the terminal impedance and the line surge impedance matrix.

\[
\text{Multiphase Reflection Coefficient} = \left[ \gamma \right] = \left( [R]^{-1} + [2]^{-1} \right)^{-1} \left( [2]^{-1} - [R]^{-1} \right)
\]  

(8.20)
It is interesting to evaluate this reflection coefficient for the simple case of a "T" conductor system where the matrix equation reduces to numerical values which can be compared with the results of Chapter 6.

\[
\begin{align*}
\text{Single Phase Case} & = \Gamma = \left( \frac{1}{R} + \frac{1}{Z} \right)^{-1} \left( \frac{1}{Z} - \frac{1}{R} \right) \\
& = \frac{1}{Z} - \frac{1}{Z} \frac{1}{R} + \frac{1}{Z} \\
& = \frac{R - Z}{R + Z} 
\end{align*}
\] (8.21)

This checks with equation (6.44). From this we see the close relationship between the single phase and the multiphase problem; more specifically we see that the single phase problem is just a special case of the multiphase problem.

IV. Multiphase Equivalent Circuit of a Line Terminal

An equivalent circuit of the line termination was developed for the single phase transmission line terminal. This single phase equivalent circuit was interpreted as the Thevenin's equivalent where the input or Thevenin's impedance was the line surge impedance and the internal or Thevenin's equivalent voltage is twice the arriving voltage. It is reasonable to expect that a similar procedure can be used for multiphase problems. In the multiphase case we must expect that the Thevenin's internal voltage will be a matrix and all impedances will be matrices also. The procedure used below is basically the same as that used in the single phase case.

The terminal voltage can be found as the sum of the forward and reverse voltage

\[
[v_e] = [v_f] + [v_r] 
\] (8.22)

Using equation (8.19) obtain
\[
[v_T] = [v_a] + \left[ [R]^{-1} + [Z]^{-1} \right]^{-1} \left[ [Z]^{-1} - [R]^{-1} \right] [v_b]
\]
\[
= \left[ [1] + [R]^{-1} + [Z]^{-1} \right]^{-1} \left[ [Z]^{-1} - [R]^{-1} \right] [v_b]
\]
\[
= \left[ [R]^{-1} + [Z]^{-1} \right]^{-1} \left[ [R]^{-1} + [Z]^{-1} - [R]^{-1} \right] [v_b]
\]
\[
[v_T] = \left[ [R]^{-1} + [Z]^{-1} \right]^{-1} \left[ [Z]^{-1} - [R]^{-1} \right] 2 [v_b]
\]  
(E.23)

Now the \([Z]^{-1}\) term can be moved inside the inverted brackets \(\left( \right)^{-1}\) if we change the order of multiplication and invert the \([Z]^{-1}\) term obtaining
\[
[v_T] = \left[ [Z] [R]^{-1} + [1] \right]^{-1} 2 [v_b]
\]  
(E.24)

Here we have a term similar to the single phase case results at least to the extent that twice the forward wave is the driving function for the equation. This equation can be checked with single phase values for comparison with the results from Chapter 6.
\[
v_T = \frac{1}{\frac{2}{R + 1}} 2 [v_b]
\]
\[
= \frac{R}{2} \frac{1}{R + 1} 2 [v_b]
\]  
(E.25)

This result compares with equation (6.45). It is not apparent that equation (E.24) can be interpreted as a matrix or multiphase voltage divider; but, because of other similarities with the single phase results, it is worth checking in a multiphase sense or equivalent circuit which is similar to the single phase equivalent circuit. Such a circuit is shown in Figure 8.03.

**FIGURE 8.03**

The voltage equation for this circuit is

$$2 \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} [2] \ + \ [R] \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(8.26)

or

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} [2] \ + \ [R] \end{bmatrix}^{-1} 2[v_x]$$

(8.27)

The terminal voltage is then found as

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = [R] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = [R] \begin{bmatrix} [2] \ + \ [R] \end{bmatrix}^{-1} 2[v_x]$$

(8.28)

Moving the [R] inside the inverted brackets requires inversion of [R] and changing the order of multiplication

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} [2] \ [R]^{-1} \ + \ [1] \end{bmatrix}^{-1} \begin{bmatrix} [2] \ [R]^{-1} \ + \ [1] \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(8.29)

But this equation for the terminal voltage of the equivalent circuit is identical to the result obtained for the end of the transmission line, equation (8.24). Thus this equivalent circuit can be used in the same manner as the single phase equivalent circuit developed in Chapter 6.

This equivalent circuit method is quite valuable for the multiphase case because this circuit can be used with unbalanced conditions such as a switch closed on one phase and open on another. It is only necessary to know the Thevenin open circuit voltage and the line surge impedance. While it has not been demonstrated here, this circuit can also be used for terminal impedances including inductances and capacitances. It is only necessary to know the system input impedance as a matrix, \( Z(s) \), where \( Z(s) \) is the Laplace transform of the input impedance of the terminal load. The general case is shown in Figure 8.04.

![Figure 8.04](image)
If the equivalent circuit is used to calculate the terminal voltage when \( [v_g] \) is known, it will be necessary to algebraically calculate the reflected voltage

\[
[v_e] = [v_g] + [v_x] \\
[v_e] = [v_g] - [v_x]
\]  

\text{(B.30)}

This again is the basic procedure used in the single phase equivalent circuit case.

**PROBLEMS**

Problem B-1 - Show that the multiphase reflection coefficient is \([-1]\) when the terminal impedance \( R \) is equal to zero on each phase. Use equation (B.20).

Solution:

The multiphase reflection coefficient, equation (B.20) is

\[
[f_2] = \left( [a]^{-1} + [2]^{-1} \right)^{-1} \left( [2]^{-1} - [a]^{-1} \right)
\]

For the condition of this problem \( [a] = [0] \) and we cannot invert the \([a]\) matrix. Therefore recognizing

\[
[a]^{-1} [a] = [1]
\]

we can write the reflection coefficient as

\[
[f_2] = \left( [a]^{-1} + [2]^{-1} \right)^{-1} [a]^{-1} [a] \left( [2]^{-1} - [a]^{-1} \right)
\]

Taking the \([a]^{-1}\) and \([a]\) inside the brackets

\[
[f_2] = \left( [1] + [a] [2]^{-1} \right)^{-1} \left( [a] \left( 2 \right)^{-1} - [1] \right)
\]

but now for \([a] = 0\)

\[
[f_2] = [1] \left( [1] \right)^{-1} = -[1] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\]

for the 3x3 case

Thus the reflection coefficient is \([-1]\).
Problem B-2 - Calculate the surge impedance matrix for a 2 conductor transmission line. Use a horizontal construction with a 30 ft separation and h = 30 ft. Use a conductor radius of 1/2 inch.

Solution:

The surge impedance matrix is given as

\[
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
= 
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
\]

\[
Z_{11} = \frac{2h}{\pi} \ln \frac{2h}{\pi} = \ln \frac{2 \times 30}{\pi} = \ln \frac{60}{\pi} = 7.27
\]

\[
Z_{12} = \frac{Z_{11}}{2} = \ln \frac{60^2}{2^2} = \ln 3.16 = 1.15
\]

Therefore

\[
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
= 
\begin{bmatrix}
7.27 & 1.15 \\
1.15 & 7.27
\end{bmatrix}
= 
\begin{bmatrix}
436 & 69 \\
69 & 436
\end{bmatrix}
\text{ ohms}
\]

Comment:

The surge impedance can be used directly in the calculation of surge problems if \([I] = [V]\) is known. That is if we know \(I_1\) and \(I_2\) the corresponding voltages \(V_1\) and \(V_2\) can be calculated.

\[
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
= 
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
\]

\[
Z_{11} \cdot I_1 + Z_{12} \cdot I_2 = V_1
\]

\[
Z_{21} \cdot I_1 + Z_{22} \cdot I_2 = V_2
\]
If the voltages are known and we wish to calculate the currents the inverse of \([Z]\) must be found.

\[
[Z]^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

The inverse of the \(2 \times 2\) surge impedance matrix can be found using the results of Problem 8-1.

\[
(Z)^{-1} = \begin{bmatrix} Z_{22} & -Z_{12} \\ -Z_{12} & Z_{11} \end{bmatrix}^{-1} = \frac{1}{Z_{11}Z_{22} - Z_{12}^2} \begin{bmatrix} Z_{22} & Z_{12} \\ Z_{12} & Z_{11} \end{bmatrix}
\]

\[
\Delta = Z_{11}Z_{22} - Z_{12}^2 = (436)^2 - (69)^2 = 191,096 - 4,761 = 186,335
\]

\[
\therefore (Z)^{-1} = \begin{bmatrix} 0.00035 & -0.000372 \\ -0.000372 & 0.000235 \end{bmatrix} \times 10^{-3} \begin{bmatrix} 2.35 & -0.372 \\ -0.372 & 2.35 \end{bmatrix}
\]

Problem 8-3 - Calculate the voltage on a 2-conductor transmission line of Problem 8-2 if the forward current on each conductor is equal to 1 amp.

Solution:

If we know the forward current, the forward voltage can be calculated using equation (8.14c).

\[
\begin{bmatrix} x_f \\ y_f \end{bmatrix} = (Z) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
\text{from the problem statement}
\]

The surge impedance matrix in Problem 8.1

\[
\begin{bmatrix} x_f \\ y_f \end{bmatrix} = \begin{bmatrix} 436 & 69 \\ 69 & 436 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 505 \\ 505 \end{bmatrix}
\]

Thus the voltage on each conductor is 505 volts and for equal current on a horizontal line the voltage
on each conductor is equal.

**Comment:**

If the above problem had been solved for a transmission line which is not of horizontal construction the voltage would not be equal on the two conductors. Note if conductor 1 were higher than conductor 2, the term \( Z_{11} \) would be greater than \( Z_{22} \) because \( h_{11} \) would be greater than \( h_{22} \). For example if the surge impedance matrix had been:

\[
\begin{bmatrix}
Z_{new}
\end{bmatrix} = \begin{bmatrix}
480 & 80 \\
80 & 440
\end{bmatrix}
\]

The voltage would have been:

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
480 & 80 \\
80 & 440
\end{bmatrix} \begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
560 \\
520
\end{bmatrix}
\]

**Problem B-6-4** - Calculate the forward voltage on the transmission line as in Problem B-6-3 but in this case consider only a current of 1 amp in conductor 1 and 0 current in conductor 2.

**Solution:**

The equations are the same as in Problem B-6-3:

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
436 & 69 \\
69 & 436
\end{bmatrix} \begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix}
436 \\
69
\end{bmatrix}
\]

We see that there is a voltage on both conductors for a current in only one conductor.

**Comment:**

Note in this problem we have not discussed just how the current flows on conductor 1 but we can see that if a forward wave is started down a line with the condition of 1 amp in conductor 1 and 0 amp in conductor 2 these current waves and the associated voltage waves \( v_1 = 436 \) and \( v_2 = 69 \) will propagate with this relationship. A sketch of the solution can be made.
This problem shows that a voltage wave propagates and would be measured on conductor 2 with a voltmeter even though the current on that conductor is zero. The voltage results from coupling or from the fact that there is current in conductor 1.

If we calculate the traveling wave energy on conductor 2 we find

\[
\text{energy} = \int v \, i \, dt = \int v \times 0 \times dt = 0
\]

so for this case there is no traveling energy on the conductor.

Problem 8-5 - Use the equivalent circuit for the terminal of a two conductor line and calculate the initial terminal current when closing a switch on conductor 1.

Solution:

The equivalent circuit for the two conductor line is

![Equivalent Circuit](image)

The current can be calculated by using the assumed 100 V source. If the circuit is quiescent the internal voltages are both 0, so we can calculate the current in conductor 1 directly. If we use the form of the equation used in Problem 8-4 we have
but if switch in conductor 2 is open the current $i_2 = 0$. So we have

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} z_{11} & 0 \\ 0 & z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

But from the problem we know $v_1$ is 100 V so from the above we can calculate $i_1$

$$v_1 = z_{11} \cdot i_1$$

$$i_1 = \frac{v_1}{z_{11}} = \frac{100}{400} = 0.2293 \text{ A}$$

Thus the current through the switch is 0.2293 A.

Comment:

From this result we can calculate the voltage on conductor 2 as

$$v_2 = z_{12} \cdot i_1 = 69 \times 0.2293 = 15.8 \text{ volts}$$

This situation is quite similar to the condition in Problem 8-4. There is a current in only one conductor and a voltage on both conductors. There will be a reflected or refracted voltage on the line that will propagate down the line following the switch closing. The wave which propagates down the line will be exactly equal to the terminal voltage. (See equation (8.30))

$$[v_x] = [v_2] - [v_1]$$

if we assume the reflected wave is in the forward direction that wave can be sketched as

\[ v_1 = 100\text{ V} \]

\[ i_1 = 0.2293\text{ A} \]

\[ v_2 = 15.8\text{ V} \]
While it may still appear unusual that a voltage can occur on a conductor when the current is zero a check can be made by multiplying the conductor voltages by \([\mathbf{Z}]^{-1}\).

\[
\begin{bmatrix}
  i_1 \\
  i_2
\end{bmatrix} = \begin{bmatrix}
  z_{11} & z_{12} \\
  z_{21} & z_{22}
\end{bmatrix}^{-1} \begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix}
\]

Using the inverse of \([\mathbf{Z}]\) from Problem 8-1 and the voltages from this problem we have

\[
\begin{bmatrix}
  i_1 \\
  i_2
\end{bmatrix} = 10^{-3} \times \begin{bmatrix}
  2.35 & -0.372 \\
  -0.372 & 7.35
\end{bmatrix} \begin{bmatrix}
  100 \\
  15.6
\end{bmatrix} = \begin{bmatrix}
  0.125 & -0.00587 \\
  -0.0372 & 0.0372
\end{bmatrix} \begin{bmatrix}
  0.2291
\end{bmatrix}
\]

This calculation checks with the current calculated above which it should because multiplying \([\mathbf{Z}]\) times \([\mathbf{I}]\) and \([\mathbf{Z}]^{-1}\) times \([\mathbf{V}]\) because these are just two ways of stating the same information.

This problem again emphasizes the effect of coupling which must be considered in multiconductor problems.

Problem 8-6 - Calculate the voltage and current on a two conductor line when closing a switch on one conductor of the line when the other conductor is grounded.

Solution:

The equivalent circuit for the problem is

![Equivalent Circuit Diagram](image)
For this problem there is no arriving voltage so

\[ V_{b1} = V_{b2} = 0 \]

For this condition we have

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= [Z]
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
\]

In this case we know both \( V_1 = V \) and \( V_2 = 0 \) so we can solve for \( i_1 \) and \( i_2 \) from

\[
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
= [Z]^{-1}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= \frac{1}{Z_{11}Z_{22} - Z_{12}Z_{21}}
\begin{bmatrix}
Z_{22} - Z_{21} \\
-Z_{12}Z_{11}
\end{bmatrix}
\begin{bmatrix}
V \\
0
\end{bmatrix}
\]

From Problem 8-2

Multiplying this equation out obtain

\[
i_1 = \frac{1}{Z_{11}Z_{22} - Z_{12}Z_{21}} (Z_{22}) V
\]

\[
i_2 = \frac{1}{Z_{11}Z_{22} - Z_{12}Z_{21}} (-Z_{12}) V
\]

Further examination of this result would show that the effective surge impedance of conductor 1 with conductor 2 shorted is

\[
i_1 = \frac{1}{Z_{11} - \frac{Z_{12}^2}{Z_{22}}}
\]

or

\[
V = Z_{\text{effective}} = Z_{11} - \frac{Z_{12}^2}{Z_{22}}
\]

Recall from Chapter 2 that this is basically the same form as the inductance with one conductor grounded.
Comment:

If we have a condition at a transmission line terminal with one conductor grounded, the effective surge impedance of the remaining conductor can be found using the same reduction as was developed for the inductive case. This is equally valid for an "m" conductor system where losses are small. In an "n" conductor system the conductors can be grounded in the same manner as described in Chapter 5.

Problem 8.7 - Use the results of Problem 8-4 as an arriving wave at an open ended line and calculate the reflected wave. Use the equivalent circuit method.

Solution:

The arriving voltage is

\[
\begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix} = \begin{bmatrix}
  436 \\
  69
\end{bmatrix}
\]

and the equivalent circuit is

\[2v_1 = 872\]

\[2v_2 = 138\]

The terminal voltage is equal to the internal voltage

\[
\begin{bmatrix}
  v_T \\
  v_2
\end{bmatrix} = \begin{bmatrix}
  436 \\
  69
\end{bmatrix} = \begin{bmatrix}
  872 \\
  138
\end{bmatrix}
\]
Equation (8.30) the reflected wave is

\[
\begin{bmatrix}
\v_r \\
\v_1 \\
\v_2
\end{bmatrix} = \begin{bmatrix}
872 \\
138 \\
69
\end{bmatrix} - \begin{bmatrix}
436 \\
69 \\
69
\end{bmatrix}
\]

Thus the reflected voltage is equal to the arriving voltage as we should expect for an open terminal.

Comment:
The reflected current is calculated using equation (8.14d)

\[
\begin{bmatrix}
\v_r \\
\v_1 \\
\v_2
\end{bmatrix} = \begin{bmatrix}2\end{bmatrix} \begin{bmatrix}1\end{bmatrix}
\]

or

\[
\begin{bmatrix}1\end{bmatrix} = \begin{bmatrix}2\end{bmatrix}^{-1} \begin{bmatrix}v_r\end{bmatrix}
\]

But the product of \([v_r]\) and \([2]^{-1}\) was found in Problem 8-5 and from that work we can conclude that the current associated with the voltage is

\[
\begin{bmatrix}1\end{bmatrix} = \begin{bmatrix}2\end{bmatrix}^{-1} \begin{bmatrix}v_r\end{bmatrix} = \begin{bmatrix}1 \\ 0 \end{bmatrix}
\]

A sketch of the solution is then

\[\begin{array}{c}
v = 430 \\
872 \\
v = 436 \\
138 \\
v = 69 \\
69
\end{array}\]

\[\begin{array}{c}
\text{CONDUCTOR 1} \\
\text{CONDUCTOR 1} \\
\text{TERMINAL} \\
\text{CONDUCTOR 1} \\
\text{CONDUCTOR 2} \\
\text{TERMINAL}
\end{array}\]

\[\begin{array}{c}
1 = 1.0 \\
1 = 0 \\
1 = 1.0 \\
\text{CURRENT = 0}
\end{array}\]
CHAPTER 9
LIGHTNING SURGES ON TRANSMISSION LINES
TRANSMISSION LINE THEORY - II

Chapter 9
LIGHTNING SURGES ON TRANSMISSION LINES

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TRANSMISSION LINE THEORY - II

CHAPTER 9
LIGHTNING STROKES ON TRANSMISSION LINES

I. INTRODUCTION

Lightning is one of the major causes of transmission line outages. The lightning performance of a transmission line is generally referred to at the number of outages per 100 miles per year of the line, where an outage is a forced interruption of the line due to a lightning stroke. The effect of a lightning stroke is to initiate an arc, either between phases or more usually from one or more phases to ground or to a grounded portion of the line structure. Depending on the instantaneous value of phase voltage, or certain characteristics of the lightning stroke itself, this arc will sometimes extinguish. In the majority of cases, however, the line voltage is sufficient to maintain the arc so initiated and the short circuit can then only be cleared by opening the line circuit breakers.

Lightning arcs may be caused by induced voltages in a line from a nearby stroke or by direct strokes to the line. For transmission insulation levels it is considered that all lightning arcs are due to direct strokes, indirect strokes giving insufficient voltage to flash over the line insulators. For distribution insulation levels this may not be true.

A number of methods for prediction of lightning performance of transmission lines are presented in the literature. These have become increasingly sophisticated with increasing transmission voltage, the older methods being found to give inaccurate answers for new types of line construction at higher voltages. For example, old methods assuming the tower can be represented as an inductance or resistance have been replaced by use of surge impedances. References 1 through 5 are in chronological sequence of the significant methods of calculation used.

The purpose of this chapter is not to review established methods or to present a new one, but rather to show how the travelling wave concept developed in the previous chapters can be used to gain a basic understanding of the lightning performance problem.

II. THE LIGHTNING STROKE

Several forms of lightning are known to exist. Basically these are cloud-to-cloud lightning, cloud-to-ground lightning, and ball lightning. There are several references 6,7 which describe the atmospheric mechanisms that create lightning stroke. We are concerned with the effects caused by lightning strokes-to-ground.

Electrically, a lightning stroke can be considered as a pulse of current. This pulse of current flows from the charged cloud through an ionized length of air (the jagged stroke seen in the sky)
through the transmission line (or tree, or whatever else it may strike) to ground. The stroke current wave shape and magnitude have a reasonably well known statistical distribution. In a recent paper, this has been shown to be a function of the height of the object struck.

The development process of a lightning stroke is much slower than the actual stroke discharge with its accompanying thunderclap. The stroke actually develops in a series of steps, each step being an ionized path, 50-100 ft. long. This stepping is of significance in shielding considerations, and causes the jagged look of lightning strokes.

Most lightning prediction methods assume that the lightning stroke can be represented by a current impulse of predetermined magnitude and wave shape with infinite internal inductance. This approach, which avoids further speculation on the characteristics of the lightning stroke, will be used in this chapter.

**Lightning Stroke Magnitudes**

Typical lightning strokes have a peak magnitude between 10 and 200 kA. The front of the current wave rises steeply to its maximum value within 1-10 μsec, typically averaging 1 μsec, decaying to half value more slowly in the order of 50-200 μsec. A lightning stroke is commonly multiple with several pulses of current discharging down the stroke channel within a few seconds. Many studies of current wave-shape and magnitude have been made and Reference 8 is the most recent summary of these.
III. USE OF TRAVELING WAVE TECHNIQUES TO CALCULATE THE EFFECT OF A LIGHTNING STROKE TO A TRANSFORMATION LINE

Stroke to Tower Top

\[ V_T = I Z_T \]

The actual value of \( V_T \) is complex and its more exact derivation is discussed later in this section.

**Stroke to Phase Conductor**

The phase voltage of a transmission line is relatively close to ground potential for a lightning stroke, and a stroke to a phase conductor can occur. The voltage to ground arising from the stroke current is then

\[ V_{cond} = I \times Z_c/2 \]

where \( Z_c \) is the surge impedance of a horizontal conductor above ground. Typically \( Z_c \) is 360 ohms. For a relatively minor lightning stroke of 10 kA we have then \( V_{cond} = 1.8 \text{ MV} \), easily enough to flash over most insulators.

To prevent this, many transmission lines use shield wires to deflect the stroke current from the phase conductor. We will now examine how this reduces the lightning voltages that cause flashover.
Effect of Shield Wires

If shield wires are present, only a portion ($I_p$) of the lightning current will go through the tower surge impedance $Z_T$ and the tower top potential is reduced to $V_T = Z_T \times I_T$. The fraction of the lightning current passing through the tower is initially dependent on the surge impedance of the tower relative to the shield wire.

The tower top potential has thus been reduced by the division of part of the stroke current.

If there are two shield wires in each direction, the equivalent surge impedance for the combination of two parallel shield wires $Z_{OH}$ is substituted for $Z_{OH}$. The commonly used approximation $Z_{OH} = Z_{OH}/2$ is valid only for wires relatively far apart (see problem 9.01).

FIGURE 9.03

FIGURE 9.04
The presence of shield wires reduces the tower top potential according to the equivalent circuits in Figure 9.04. The voltage across the insulator string is proportional to the tower top voltage and therefore is similarly reduced.

**Effect of Footing Resistance**

Although a transmission tower has foundations in the soil, it has a finite footing resistance or resistance to a remote ground due to the resistivity of the soil. Typical footing resistances are in the 1-50 ohm range.

The voltage wave \( V_T = I_T Z_T \) travels down the tower to the base. At the base it sees a footing resistance \( Z_G \) and is reflected with a reflection coefficient which from traveling wave theory is

\[
K = \frac{Z_G - Z_T}{Z_G + Z_T}
\]

(9.01)

As \( Z_G \) is usually less than \( Z_T \), \( K \) is negative and this reflected voltage wave travels back to the tower top and lowers the tower top potential.

For short towers the voltage waves traveling up and down the tower quickly reduce the tower top potential to the value

\[
V_T = I_T \times Z_G
\]

In many lightning calculations, e.g., those considering short towers or strokes with long rise times, the tower surge impedance may be neglected.

**Effect of Adjacent Towers**

Using the traveling wave concept it is apparent that the tower top potential would be influenced by reflected waves first from the adjacent towers and later from more distant towers.

\[
\begin{align*}
&Z \\
&\text{I} \\
&Z_T \\
&Z_G \\
&\text{V}_0 \\
&Z_T \\
&Z_G \\
&\text{V}_0 \\
&Z_T \\
&Z_G \\
&Z 
\end{align*}
\]

**FIGURE 9.05(a)**
Figure 9.05(a) shows a central tower struck by lightning with adjacent towers equidistant on either side. The time taken for the first traveling waves $V_g$ to reach the adjacent towers is $\Delta t$; $\Delta t$ being given by

$$\Delta t = \frac{\text{Span}}{c} \quad (9.02)$$

where $c$ is the velocity of the traveling waves; in this case, approximately the velocity of light.

If we ignore $Z_T$ (for simplicity), up to the time $2\Delta t$, the tower top potential equals $V_T T$, where $V_T$ is the tower current. However, at $2\Delta t$ the reflected waves from the adjacent towers arrive. Reflections from the second closest tower on both sides of the tower struck by lightning will influence the tower voltages at time $4\Delta t$ and so on. The reflection coefficient at each tower is indicated in Figure 9.05(b). The maximum value as well as the wave shape of the tower top potential will be dependent on the relation between $\Delta t$ and the front time of the lightning current.

For a 1000 ft. span, $\Delta t = 1$ usec

$$Z_{BH} = 400 \text{ ohms}$$
$$Z_G = 20$$
$$V_g = \frac{20 \times 200}{20 + 200} = 18.21$$

The reflection coefficient at each tower top

$$k = \frac{20 \times 400 - 400}{20 + 400 + 400} = -0.91$$

![Diagram showing voltage relationships at different times](image)
Figure 9.05(b) shows the lattice diagram that can be developed from this. Two examples based on Figure 9.05(a) will illustrate the effect of reflections.

The following two example cases illustrate the technique. Figure 9.05(c) shows a simplified stroke wave form used in these problems.

![Diagram](image)

**FIGURE 9.05(c)**

**CASE 1**  
**Front Time** $T = 0.1$ sec.

From Figure 9.05(b) the initial voltage at the tower top is $V_B = 18.2 \times 1$; i.e., a ramp voltage that increases with $18.2 \times 50 = 910$ kV/sec. The tower top potential $V_B$ will increase from zero to 910 kV in 1 usec as shown in Fig. 9.06 and then stay constant up to time 2 usec [207] when reflections arrive from the adjacent towers on both sides. From the lattice diagram in Figure 9.05(b), it is seen that each incoming wave has a voltage of $V_B$ which translates into a contribution to $V_T$ of $(1-K)V_B$ for each of the two waves. Since the lightning current at this time (2 usec) has reached its constant value, the slope of $V_T$ will change to

$$2 \times (1 + K)V_B = 2(1 - 0.91)/(-0.91) \times 18.2 \times 50 \text{ kV/μsec}.$$  

$$= -149 \text{ kV/μsec}.$$  

The waves reflected from the center tower at time 2 usec has a magnitude of $-(1-1.82), 91$ $\times$ 910 kV/μsec. They return at time 4 usec., their magnitude reduced to $-(1-1.82), 91^2 \times 910$ kV/μsec, and cause a change in $V_T$ of $2(1-0.91)^2(1-1.82), 91^2 \times 910 = -111.2$ kV/μsec. At time 4 usec, there is also a contribution to $V_T$ from the first reflection from the second towers in each direction. The magnitude of this contribution is $2 \times (1-0.91)^2(1-1.82), 91 \times 910 = -1.2$ kV/μsec. With constant lightning current and no losses in the ground wires, $V_T$ will decrease towards zero in the pattern shown in Figure 9.06.

In connection with this analysis it is interesting to note how fast the initial lightning surge attenuates from one span to the next. The span closest to the tower struck by lightning experiences a maximum initial surge voltage of 910 kV. On the second span the corresponding maximum voltage is $(1-0.91) \times 910 = 0.09 \times 910 = 82$ kV. On the third span 0.09 x 82 = 7.4 kV and so on. Because of this rapid attenuation it is generally sufficient to include only the effect of the two adjacent towers when analyzing the tower top potential for the first few micro-seconds.
Case 2 - Front Time $T = 3 \text{ usec}$.

In this case the initial voltage surge $V_0 = 50/3 \times 18.2 \text{ kV/usec.} = 303 \text{ kV/usec.}$ and the tower top potential as a function of time develops as shown in Figure 9.07. It is observed that for front times longer than two span travel times the maximum tower top potential will be less than that computed without reflections.

**Strokes to Shield Wire**

More lightning strokes are likely to hit the shield wire at some distance from the tower than at the tower itself. In Figure 9.08 a lightning stroke has terminated on the shield wire one-third span length from tower A. The lightning current $I$ rises to $50 \text{ kAmps.}$ in 1 usec. Using the lattice diagram where $V_0 = 50 \times 200 = 10,000 \text{ kV/usec.}$ and $K = 0.91$, the tower top potential at A develops as shown. The oscillatory nature of the voltage may be explained physically if one recognizes that the piece
of line from the lightning stroke termination to Tower A acts as an inductance. In terms of the
lattice diagram, the incoming controlling waves are of a sequence \( V_0, \alpha V_0, \alpha^2 V_0, \alpha^3 V_0, \ldots \), i.e.,
alternately negative and positive waves resulting in an oscillatory attenuating voltage of fre-
quency:

\[
F = \frac{1}{2 \pi \text{sec}} = 300,000 \text{ Hz}
\]

In contrast, the incoming waves for strokes directly to the tower had a sequence \( \alpha V_0, \alpha^2 V_0, \alpha^3 V_0, \ldots \),
i.e., all negative resulting in an asymptotic attenuating voltage.
IV. TOWER REPRESENTATION

In the foregoing examples the tower was assumed to be of zero impedance and having zero travel time. The tower top potential which equaled the footing potential was due to current through the soil represented by the equivalent “footing resistance.”

Figure 9.08 illustrates a widely used tower representation. The tower itself is represented by an equivalent conductor with surge impedance dependent on tower dimensions and travel time $T = h/C$ where $h$ is the tower height and $C$ the speed of light. Although the accuracy of the model is questioned by many researchers, it facilitates the use of traveling wave concepts in explaining the effect of tower type and tower height on lightning performance. The proper surge impedance for the equivalent conductor can be assessed for simple tower configurations by solving the electric and magnetic field equations.

![Figure 9.08](image)

For a cylindrical tower of radius $R_T$ and height $h$, Wagner and Heileman (Ref. 4) proposes the use of the following equivalent surge impedance for strokes directly to the tower top.

$$Z_T = 60 \ln(200R_T/h)$$  \hspace{1cm} (9.05)

Although the expression may appear similar to the surge impedance of a horizontal conductor above ground, the differences are substantial. For example, this equivalent surge impedance increases with the length of the “conductor.” The phrase “equivalent surge impedance” is used because the actual surge impedance defined by $Z_{GC}$ is not the same at all points of the tower, but will decrease from top to bottom as the capacitance to ground per unit length increases.

Many attempts have been made to measure this parameter. Results are given in Reference 10 that suggest representing a tower by an inverted cone, while Jordan 12 in 1937 suggested a formula that has been found to be valid even today.

Referring to Figure 9.08, a lightning strike will result in a voltage surge of $I \times Z_T$ traveling down the tower, reaching the footing resistance $Z_F$ after a time $\Delta T$. At time $2\Delta T$ a reflected wave will
reach the tower top and (provided $Z_o < Z_f$) reduce $V_I$. The effect of the tower then is to delay the beneficial effect of a low footing resistance - a delay that is more serious the steeper the current wave-front is (high $dI/dt$) and the higher the tower. In this regard the tower acts much like an inductance. (In older lightning prediction methods, the tower is represented as a lumped inductance.)

Then, what is the effect of the tower? How does the tower top potential vary with tower height and tower type? To investigate the effect of height, consider the situation in Figure 9.10.

![Diagram](attachment:Figure_9.10.png)
From equation (9.03):
\[ Z_f = 60 \text{ km} \Omega \frac{2 \times 100}{10} = 200 \text{ ohms} \]

With a 410 ohm surge impedance for the shield wire, this yields an impedance match at the tower top \((Z_0 = 0)\). The tower top voltage develops as shown. \((V_g = 5000 \text{kV/\musec})\). Neglecting the tower effect, the tower top potential would have been rising with 910 \text{kV/\musec} as shown (dotted line). Note that from \(T = 2\pi f < 0.2 \text{ \musec}\) and for as long as the current is steadily increasing \((dI/dt \text{ constant})\), the effect of the tower is represented by a constant voltage,

\[ V_T = (1000 - 910 \times 0.2) = 818 \text{ kV} \]

With zero footing resistance, this voltage would have maintained a constant value \((1000 \text{ kV})\) after 0.2 \text{ \musec}. For \(Z_0 = 2000\), \(V_T\) would have increased at the rate of 5000 \text{kV/\musec}, past 0.2 \text{ \musec}, and for \(Z_0 < \) the voltage would have increased at 10000 \text{kV/\musec} after 0.2 \text{ \musec}.

The choice of \(Z_f = \frac{Z_0}{2}\) has made the calculations simple by reducing the number of reflections.

From this discussion the following observations can be made:
1) The tower top potential increases with tower height because of increased equivalent tower surge impedance as well as tower travel time.
2) The tower top potential decreases with increased tower radius.
3) The voltage increase due to the tower is proportional to the rate of rise of the lightning current.
4) The relative influence of the tower is greater for low footing resistance.

The effects of varying the tower height, footing resistance, and span length are analyzed in detail in Reference 13.

It may be noted in closing this section that the above conclusions would be equally apparent if the tower was represented by an inductance. In particular, the physical inductance of the tower will increase with tower height and decrease with increased tower radius. Both the surge impedance approach and the inductance representation are valuable models in judging the relative lightning performance of various tower types, but it should be emphasized that they are limited representations of the complex electric and magnetic fields around the tower struck by lightning.
V. VOLTAGES ACROSS THE INSULATOR STRING

The ultimate purpose of lightning surge computations is to estimate the voltages across insulator strings, knowing the insulation strength (flashover voltage) the possibility of flashover and line tripout can be evaluated.

\[
\begin{align*}
V_I &= V_C - V_B = V_T - V_B = V_T(1-c) \\
\end{align*}
\]

where \( c \) is the shield-phase conductor coupling factor. Therefore, the lightning performance of a line can be improved by: 1) either lowering \( V_B \) (smaller \( I_b \) shorter and thicker towers), or 2) by increasing the coupling between shield wire and phase wire (bringing them closer together). The coupling from the shield wire to each of the three phases is given by the surge impedance matrix equation:

\[
\begin{bmatrix}
V_A \\
V_B \\
V_C \\
V_D \\
\end{bmatrix} = \begin{bmatrix}
Z & 0 \\
0 & Z \\
0 & Z \\
0 & Z \\
\end{bmatrix} \begin{bmatrix}
I_A \\
I_B \\
I_C \\
I_D \\
\end{bmatrix}
\]

\[(9.04)\]
where \( V_p \), \( V_y \), and \( V_g \) are voltage surges on the phase conductor due to the current surge on the shield wire and \( V_g \) is the voltage surge on the shield wire. Considering only one phase conductor as in Figure 9.12, the equation is (Chapter 8):

\[
\begin{bmatrix}
V_p \\
V_y \\
V_g
\end{bmatrix} = 60 \begin{bmatrix}
\ln \frac{d}{R_p} & \ln \frac{d}{R_y} & \ln \frac{d}{R_g}
\end{bmatrix} \begin{bmatrix}
0 \\
I_g
\end{bmatrix}
\]

This is equation (9.05).

That the current in the phase conductor is zero for a lightning stroke to the shield wire is apparent when considering the symmetry of the problem. If the voltage surges to the left and to the right on the phase wire were accompanied by currents \( I_g \), these currents would have to be equal in magnitude and opposite in direction. The only current magnitude satisfying both of these conditions is zero. But \( I_g \), of course, would equal 1/2. Then from equation (9.05)

\[
\begin{align*}
V_p &= \frac{1}{2} \cdot 60 \ln \frac{d}{d/d} \\
V_y &= \frac{1}{2} \cdot 60 \ln \frac{2R_y}{R_y}
\end{align*}
\]

or

\[
V_g = \frac{\ln \frac{d}{R_y}}{\ln \frac{d}{R_g}} \cdot V_y + c V_y
\]

Equation (9.06)

The coupling factor \( c = \frac{\ln \frac{d}{R_y}}{\ln \frac{d}{R_g}} \) will typically have a value of 0.2 to 0.3.

In a vertical phase configuration as shown in Figure 9.11, one would think then, that the insulator supporting the lowest phase conductor, (having the lowest coupling factor to the shield wire)
would be exposed to the highest lightning surges. There is, however, a voltage drop down the tower which causes a lower cross arm voltage ($V_c$) at the lower phase. Usually the tower effect dominates the tower top potential (low footing resistance and steep wavefront) and the voltage drop from the highest cross arm to the lowest cross arm exceeds the voltage difference in coupled voltage, making the surge voltage across the insulator ($V_c - V_d$) higher for the upper phases. Both the surge impedance and the inductance representations of the tower effect can be used for numerical evaluations of the voltage profile along the tower.

**Surge Voltage Measurements on Models**

An alternative to surge computations using the traveling wave concepts and mathematical models of the tower is to build a scale model of the transmission line and measure the voltage across insulators resulting from lightning type currents injected at the tower top or some place on the shield wire. Neglecting possible nonlinear effects (corona losses) such models correctly represent the electrostatic and electromagnetic field.

There are several measurement problems due to the high frequencies and the accuracy of the stroke representation. The lightning performance estimation method presented in Reference 3 is based on such scale model measurements.

Generally, linear measurements on models are used simply to calibrate computer programs for lightning performance calculations.

VI. **SHIELDING FAILURES (DIRECT STROKES TO PHASE CONDUCTORS)**

Strokes to a line may strike the phase conductor, the shield wire (if there is one) or the tower itself. Strokes to a phase conductor, when there is a shield wire, cause outages known as shielding failures.

**FIGURE 9.13**

In an earlier section we have seen that the insulator voltages per unit of stroke current are much greater than for strokes to the shield.
A high proportion of lightning outages are now believed due to shielding failures, and the results of a very extensive study are given in Reference 14.

A useful rule of thumb is that a tall object can be expected to shield lower objects within a 30 degree cone of protection from the highest point. For shield wires this can be extended to mean that phase conductors parallel to a shield wire and within this 30 degree cone will be protected from lightning. In practice this is not always true, and Reference 14 allows many refinements to the design of shield wires.

VII. OTHER CONSIDERATIONS

The impulse voltages that develop on the line due to the stroke are a complex function of the stroke itself, the current pulse magnitude and shape, the size and height of the various line components, the exposure of the line or the shielding by trees or terrain, the line insulation, and the line structure resistance to ground ( footing resistance).

Factors that we have not considered here are the effects of induction to the line from the body of the stroke itself, the effects of pre-discharge currents that may develop between parallel conductors prior to flashover that alter the electrostatic induction, and non-linear effects such as the effective increase in size of conductors due to corona when they are at high potentials. All of these components plus those we have considered in detail are components used to make up a lightning performance calculation.

Computer programs that calculate outage rates often use the 'Monte Carlo' technique. This is a method that uses repeated calculations to obtain the statistical performance of a line, and can be used to greatly simplify calculations although it has the disadvantage that computer time is greatly increased. References 15 and 16 detail use of this method.

Other readings of interest are References 17 - 19.
REFERENCES


PROBLEMS

Problem 9-1 - Compute the equivalent surge impedance \( Z_1 \) for two parallel shield wires as a function of their separation \( D \). The shield wires are 50 feet above ground and have a radius of 0.25".

![Diagram showing two parallel shield wires with labels and equations.]

**Solution:**

The matrix equation relating surge voltages to surge currents is

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]  

(9.07)

where

\[
Z_{11} = Z_{22} = 60 \log \frac{2 \times 50 \times 12}{0.25} = 500 \text{ ohms}
\]

\[
Z_{12} = Z_{21} = 60 \log \frac{D}{(2 \times 50)^2}
\]

Referring to Figure 9.14(a) the equivalent surge impedance \( Z_1 \) equals the ratio \( V_1/I_1 \) where \( V_1 = V_1 = V_2 \) and \( I_1 = I_1 + I_2 \) in the physical configuration. Replacing the expression for \( V_2 \) in equation (9.07) by \( V_2 - V_1 \) one has:

\[
\begin{bmatrix}
V_1 \\
V_2 - V_1
\end{bmatrix} = \begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} - Z_{11} & Z_{22} - Z_{12}
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]  

(9.08)

Similarly the \( Z \) matrix can be modified to fit a new current vector where the first element is \((I_1 + I_2)\):

\[
\begin{bmatrix}
V_1 \\
V_2 - V_1
\end{bmatrix} = \begin{bmatrix}
Z_{11} & (Z_{12} - Z_{11}) \\
(Z_{21} - Z_{11}) & (Z_{22} - Z_{12}) - Z_{11}
\end{bmatrix} \begin{bmatrix}
I_1 + I_2 \\
I_2
\end{bmatrix}
\]
Since $V_2 - V_1 = 0$, $i_2$ can be eliminated:

$$i_2 = \frac{Z_{21} - Z_{11}}{Z_{11} + Z_{22}} (i_1 + i_2)$$

Then:

$$V_1 = \left( Z_{11} - \frac{(Z_{22} - Z_{11})^2}{Z_{11} + Z_{22}} \right) (i_1 + i_2)$$

Since $i^1 = V_1$ and $i^1 = i_1 + i_2$, the equivalent surge impedance for two parallel shield wires is:

$$Z^1 = Z_{11} - \frac{(Z_{22} - Z_{11})^2}{Z_{11} + Z_{22}}$$

In this particular problem $Z_{11} = Z_{22} = Z$

$$Z^1 = Z_{11} - \frac{(Z_{22} - Z_{11})^2}{2Z_{11}} = Z_{11} - \frac{Z_{22} - Z_{11}}{2}$$

$$Z^1 = \frac{(Z_{22} - Z_{11})}{2}$$

$$= \frac{508 + 60 \ln \frac{D+100}{D}}{2}$$

Since $\lim_{D \to 0} \left( \frac{D+100}{D} \right)$, it is seen that $Z^1$ approaches $Z/2$ when $D$ becomes large.

<table>
<thead>
<tr>
<th>D (Feet)</th>
<th>$D+100$</th>
<th>$D$</th>
<th>$\ln \frac{D+100}{D}$</th>
<th>$Z^1$</th>
<th>$Z^1/Z$</th>
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<td>278</td>
<td>.55</td>
</tr>
<tr>
<td>100</td>
<td>1.41</td>
<td>100</td>
<td>2.3021</td>
<td>264</td>
<td>.52</td>
</tr>
</tbody>
</table>

The above results indicate that coupling between shield wires should always be considered.

Note that the mathematical technique used here is identical to that used for conductor bundling in Chapter 5.
Problem 9-2 - Compute the effect of one and two shield wires in reducing the tower top potential \( V_T \) for strokes to the tower top. Neglect the tower effect and reflections from adjacent towers. Referring to Figure 9.04, the footing resistance \( R_f = 10 \Omega \). The shield wires have a radius of 0.25" and are positioned 50 feet above ground as in Problem 9-1. For the 2-wire case, the separation is 20 feet. The maximum lightning current is 50 kA.

Solution:

Case 1 - No Shield Wires

\[ V_T = 10 \times I \]
\[ V_{T MAX} = 500 \text{ kV} \]

Case 2 - 1 Shield Wire

\[ Z = \frac{Z_0}{2} = 254 \text{ ohms} \]
\[ V_T = \frac{Z_0}{Z_0 + Z} \times Z \times I \]
\[ V_{T MAX} = 481 \text{ kV} \]

Case 3 - 2 Shield Wires

\[ Z = \frac{Z_0}{2} = 151.5 \text{ ohms} \]
\[ V_T = \frac{Z_0}{Z_0 + Z} \times Z \times I \]
\[ V_{T MAX} = 469 \text{ kV} \]

Comment:

In this simplified example the shield wires do not greatly affect the tower top voltage. Where tower surge impedance is considered it may be shown that the effect is much more significant, and this is recommended as an exercise for the student. The most significant effect of the shield wires is their coupling to the phase conductor reducing the potential difference over the insulator strings. The reflections from adjacent structures help reduce the tower top voltage more quickly, and the reduced total impedance to ground from all the towers in parallel that reduces the tower top potential at longer times after the stroke occurs.
APPENDIX 9A

TYPICAL LIGHTNING CALCULATIONS
TRANSMISSION LINE THEORY II

APPENDIX D

SOME TYPICAL LIGHTNING CALCULATIONS

INTRODUCTION

The PTI Lightning Tripout Program calculates the single-and/or double-circuit outage rate of a given line design using a wave table technique for any assumed front time, waveshape, insulator performance, shielding failure assumptions, or component representation.

This program has been used to calculate the following results to illustrate the significance of some line design parameters. It is emphasized that the results shown are comparative only and do not necessarily represent realistic line performance, determination of which must include many local considerations such as grounding, shielding and varying exposure and terrain.

PARAMETERS

A. Effect of Line Insulation

As illustrated in Figure 1, for a typical line the outage rate varies quite substantially with the insulation BIL.

Adding discs to improve lightning performance in selected areas, e.g., in exposed regions, is not uncommon but has the disadvantage of requiring additional tower clearances for coordinated insulation.

![Figure 1](image)

Effect of BIL on Lightning Performance
B. Effect of Footing Resistance (Z_g)

This is perhaps the most significant parameter in determining lightning performance. Depending on the line insulation used, there is usually a critical value of Z_g below which the tripout rate is effectively zero; for a 345 kV line this is often selected as 20-30 ohms, with lower values at lower voltages and, hence, BIL's. Figure 2 shows this dramatic change in line performance as a function of Z_g.

C. Shield Wires

As illustrated in Figure 2, the addition of a shield wire gives a substantial improvement in lightning performance due to the diversion of strokes from the phase conductor and the discharge of stroke to ground through the relatively low impedance of the shield wire and tower, rather than through the high impedance of a phase conductor.

The illustration shows a further improvement in performance with two shield wires. This improvement is more marked where the two shield wires are widely spaced, and the tower is tall and slender.

A third shield wire is often considered, strung between the phase conductors rather than above. It is not strictly a shield wire as it does not protect the phase conductors from direct

---

**FIGURE 2**
Effect of Footing Resistance and Shielding on Tripout Rate
strokes; rather it provides additional coupling of impulse voltages to the phase conductors, so reducing the impulse voltages over the line insulation. Quite substantial improvements in lightning performance have been calculated for this technique, but no major installations of it are known.

D. Tower Dimensions

Generally the shape and configuration of the tower is ignored for single circuit lines. The important variables are the tower height and effective radius.

As an initial approximation, the surge impedance $Z_T$ of a tower is given by Jordan's formula:

$$Z_T = 600 \ln \frac{d}{b} + 90 \frac{c}{b} - 60 \text{ ohms}.$$  

From this it is seen that variations in $r$ and $h$, often interdependent, produce little significant variation in $Z_T$. However, increase of the tower height increases the travel time of voltage waves up and down the tower, and this has a significant effect as shown in Figure 3.

Figure 4 illustrates that the radius alone has little significance in determining the tripout rate.

Higher towers are also subject to an increase in tripout rates due to their greater exposure to lightning and to an increased probability of shielding failures. It has also been suggested that

```
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tower_height_tripout_rate.png}
\caption{Effect of Tower Height on Tripout Rate}
\end{figure}
```
strokes to higher towers are more severe than strokes to shorter structures.

E. Span

As illustrated in Figure 5, longer spans will give higher tripout rates. Longer span lengths increase the tripout rate in two ways, firstly by increasing the time for voltage reflections from adjacent towers, and secondly by increasing the likelihood of a midspan flashover from shield wires to phase conductors following a midspan stroke. This second effect is often difficult to factor in, as exposure of conductors and, hence, the relative probability of a midspan stroke, varies widely with terrain.

F. Shielding Angle

The shielding angle of a shield wire is measured as the vertical angle from it to the outermost phase conductor, and is conventionally positive for phase conductors outboard of the shield wire and negative for the reverse.

A complex relationship has been developed for determining shielding failures. Figure 6, derived from the Pathfinder results of Whitnemed, illustrates some of the considerations.
Figure 6
Effect of Span Length on Tripout Rate

Figure 6
Critical Shielding Angle Curves
DOUBLE-CIRCUIT TRIPOUTS

On multi-circuit lines the double-circuit tripout rate is often of great concern. This is where a single lightning stroke causes a tripout on more than one circuit. Calculation of double-circuit tripouts is complex, as the sparkover of any one phase provides an additional shunt current path and changes both the coupling to the remaining phases and the tower voltage.

Rearrangement of the phases on the tower or the use of unbalanced insulation to provide a sacrificial circuit have been calculated to have some moderate effect on the double-circuit tripout rate but do not provide major improvements in performance.

Typical curves showing the effects of tower height, insulation and footing resistance on a 500 kV double-circuit line are given in Figures 7 - 9.

![Diagram](Image)

**FIGURE 7**
Effect of Tower Height on Tripout Rate
FIGURE B.
Effect of Insulation on Tripout Rate
GROUNDING

As discussed for Figure 10, there is often a clear dividing line for footing resistance above which performance declines rapidly. Reduction of lightning tripouts by reducing footing resistance usually is economically attractive, and reduced footing resistance is relatively simple to achieve. However, it is possible to add large amounts of grounding to a line (ineffectively), and these paragraphs discuss both good grounding design and the best methods of installation.

Grounding Design

Reference 1 gives a detailed analysis of resistance to ground of buried electrodes. Generally the best grounding is obtained by using widely dispersed electrodes; installation of large numbers of electrodes within a small area provides very little reduction of resistance to ground.

Figure 10 on the next page shows a typical relationship of footing resistance with depth for vertical rod electrodes.

FIGURE 10
Resistance With Depth of Multiple Electrodes

Note that even with the large spacings from a typical tower assumed, less improvement is obtained with each successive electrode.

Another consideration for grounding is the surge impedance of the grounds and their connecting conductors. The surge impedance of a multiple legged cross foot is shown in Figure 11.

FIGURE 11
Effect of Number of Wires on the Counterpoise Impedance
Note that there is a distinct advantage to using four grounding paths rather than one, even if the leakage resistance is identical. It must also be borne in mind that the shorter the connection to the ground electrode, the faster will be the voltage decrease from the surge current to the leakage current value. In fact, a single long counterpoise will have a lower leakage resistance than a crows foot arrangement but will have a higher transient impedance.

Grounding design is thus a compromise between the requirements for low leakage resistance and low and fast decaying surge impedance.

Typical Grounding - Improvement

This is a function of the soil and the degree of improvement needed.

In stone-free soil with high conductivity, galvanized steel driven stakes approximately 9 ft. long will usually suffice. These stakes are paralleled 18 inches below ground level with a galvanized steel strap or conductor.

In soil with low conductivity, long electrodes may be needed to reach a high conductivity strata. Driven rods with tapered or threaded connections can be easily installed to depths of 150 feet with a pneumatic hammer.

In rocky soil, driving electrodes is often impossible. It is best to drill a hole and backfill around an electrode. Usually the low conductivity requires that bored holes be at least 20 feet deep; and depending on the technique used, electrodes can readily be installed in this manner to depths of 200 feet.

Counterpoise is often propounded as a method of grounding improvement. This term is used when relatively long lengths of electrode are run radially from the tower or along the right-of-way parallel to the line, buried horizontally at a shallow depth (say, 6-18 inches). Compared to the methods above, it has disadvantages, as although multiple counterpoise can provide a low surge impedance, a low leakage impedance is often not obtained within a reasonable distance of the tower, and the effective impedance to a lightning stroke is, therefore, higher than it should be. Continuous counterpoise running from one tower to the next is particularly unsatisfactory in this respect. Reference 2 provides greater detail of grounding materials and techniques.

Cost Control of Grounding Additions

Since tower sites are irregular and soil varies with location and depth, a wide range of grounding may be needed to provide desired footing resistances. It is obviously wasteful to add a maximum design to all structures; also in some locations a maximum installation may be inadequate.

(2) I. Grant, W. Purcell, "Practical Grounding of Transmission Lines," IEEE Paper 71CP 166 PWR.
Measurement of footing resistance of towers prior to connection of the shield wire, or of soil resistivity in the tower location, or prior to the tower erection, will provide an estimate of the installation needed to reach a desired footing resistance. This can be used to prepare an optimized design for each location and for material, cost, and time estimates.

As installation of additional grounding proceeds for each tower, measurements can be taken and the installation curtailed or extended depending on the results from each individual site.

Measurement of Grounding Resistance and Soil Resistivity

The three-electrode method is usually used to measure tower footing resistance. A low voltage, low current supply is connected to the tower as shown in Figure 12.

![Figure 12: Three-Electrode Method for Resistance Measurement](image)

The ratio of voltage and current then provides the impedance to ground of the tower base. Note that the shield wires must be disconnected at the tower top to prevent parallel measurements of adjacent tower bases.

The distance from the electrode to each other and the tower is a function of the tower dimensions and the soil. Sufficient spacing must be allowed to avoid local effects. Typical spacings would be L = 350 ft. between the tower and electrodes.

Measurement of large grounds such as substations may require spacings of 1/2 mile or more. Often with large grounds, however, the value is too low to be measured accurately this way and an alternative is used with a high current source such as using a grounded 1-phase transformer and returning current along a transmission line from a remote ground, with voltage measured to a true remote ground obtained through a telephone line.

Soil resistivity is measured with either a three- or four-electrode method, as before by taking the ratio between voltage and injected current and then deriving the soil resistivity, assumed homogeneous to a depth equal to the electrode spacing.
For low current measurements of the three- or four-electrode technique, it is preferable to use a Megger or special device such as a Geoscan Geohm, with injection current frequencies that are not a multiple of supply voltage frequency and, therefore, not subject to error from ground currents.

**Typical Values**

1) Typical soil resistivities are:
   - 10- 30 ohms in swamp
   - 10- 60 ohms in clay
   - 400- 600 ohms in loam
   - 700-3000 ohms in sand and sandstone

2) Typical tower footing resistances as a function of resistivity are:
   - Steel tower, grillage foundation \( r_g \cdot \text{soil resistivity} \)
   - Wood pole, H-frame \( r_g \cdot \text{soil resistivity} \)

**Other Considerations**

The high current resistance of a ground electrode may be substantially less than its low current resistance due to voltage breakdown in the soil. General practice is to design for the low current resistance as measured, however, as this gives a safety margin and an allowance for resistance variation with seasonal changes in soil moisture and, hence, conductivity.

**FUTURE DESIGN TECHNIQUES**

Recently techniques have been developed to permit optimized design of ground electrode systems. These are both analytic and experimental. Reference 3 describes a recent analytic approach to grounding design, and Reference 4 describes the use of modeling using an electrolytic pool to model the soil. Future grounding designs may be further improved by application of these techniques.

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CHAPTER 10

IMPULSE OVERVOLTAGES - TERMINALS
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TRANSMISSION LINE THEORY-II

CHAPTER 10
IMPULSE OVERVOLTAGES - TERMINALS

I. INTRODUCTION

The fundamentals of traveling waves have been discussed in prior chapters. These fundamentals have been applied to the evaluation of the impulse response or lightning performance of overhead lines in Chapter 9. In this chapter we will briefly consider the response of terminal equipment, namely transformer windings, as well as response of the terminal transmission line network as this relates to lightning arrester separation evaluation.

II. TRANSFORMER WINDING ANALYSIS

The transformer winding should be considered for the surge analysis both to determine how this winding influences the external circuit and also how a surge stresses the winding itself. A complete analysis of an actual transformer is so complex that a detailed analysis of each design is not made. Rather, simplified models are analyzed and results from such analysis can be used both by the manufacturer in the design of the transformer and by application engineers relative to the two areas of consideration above; i.e., influence on the system and stresses within the transformer itself.

The purpose of the analysis in this section is only intended to be introductory with the intention of presenting only the most fundamental concepts. Further reference can be made to texts dedicated to this subject.¹,²

The problem of surges in transformer windings is quite complex and often must be evaluated in the final production transformer. The tests through which this is achieved are developed and revised by standardizing committees and will be discussed in the course on Insulation Coordination. Design evaluation has been performed in full scale models designed with special monitoring points. These transformers can be tested to failure to completely evaluate design calculations and procedures. Because this is a very expensive procedure, model³ and analytical⁴ techniques have been developed to evaluate transformer design. Primary emphasis in this chapter will be placed on the fundamentals of the analytical approach.

A. Single Layer Coil Differential Equation

The analysis of a single layer coil can be performed without great difficulty if certain simplifying assumptions can be made regarding the physical characteristics of the problem. While this problem presents the most fundamental solution of the surge distribution in windings, considerable insight can be obtained from this solution. In addition, this configuration is common in core form
reactors and transformers utilizing barrel or layer coils. In such a winding as sketched in Figure 10.01(a) it should be observed that the disturbance propagation turn-to-turn through the insulation is very much faster than the disturbance propagation along the conductor, the ratio of turn-to-turn propagation to conductor propagation being on the order of the ratio of twice the conductor insulation thickness plus the gap between turns to the length of the turn. This leads to disturbance propagation time turn-to-turn of the order of 0.01 microseconds while the propagation time along a 100 turn coil with a mean radius of 30 inches would be about 1 microsecond depending on the effective permeability of the structure.

![Diagram](image)

**FIGURE 10.01**

Although the magnetic and electric fields set up by disturbances in a coil are rather complex, an important simplification yields a useful equivalent for the layer coil. The equivalent may be formed if the following assumptions are accepted:

a) Adjacent turns serve as shields to both magnetic and electric fields. That is, all electric and magnetic coupling may be described by the coupling between adjacent turns. For example, consider three turns; the middle turn is completely shielded by the two adjacent turns.\(^1\)

---

1. \(\star\) No inductance nor capacitance between non-adjacent turns in equivalent circuit.
b) All other conducting surfaces which are capacitively coupled to each turn in the
coil are assumed to be at ground potential, and further, mutual inductive coupling
existing between the grounded structures and each turn is neglected. Under these assumptions the equivalent circuit shown in Figure 10.01(b) applies. Ordinarily the equivalent is utilized to describe the magnetic and electric coupling between successive turns and external electrodes. To facilitate analysis it is common to approximate the turn equivalent circuit by considering the inductance and capacitance associated per unit length along the turn; taking this association to the limit of the differential element of length, the equivalent then appears as shown in Figure 10.02.

\[
\begin{align*}
\frac{dv}{dt} & = Ldx \\
\frac{di}{dt} & = v \\
\frac{dx}{dt} & = \frac{v}{K} \\
\frac{dv}{dt} & = \frac{B}{L} i \\
\end{align*}
\]

**FIGURE 10.02**

Note that this equivalent circuit would be identical to that for an overhead line if \( K \) were not included. But the inter-turn capacitance between windings \( K \) on the above diagram is of a great enough magnitude that the physical result is different than the transmission line type traveling wave. Some analysis has been published which analyzes transformer transients from a traveling wave point of view but the majority of the papers and texts use a standing wave method of analysis used in these notes. Three differential equations can be written from this circuit:

1. T.I.E., no inductance in equivalent circuit to ground.
\[ \frac{d}{dx} \left( \frac{1}{x} \frac{dv}{dx} \right) - C \frac{dv}{dx} = 0 \quad (10.01) \]

\[ \frac{v}{x} = \frac{1}{L} \frac{dv}{dx} \quad (10.02) \]

\[ I_k = -K \frac{s^2}{s^2x} \quad (10.03) \]

where

- \( L \) = inductance per unit length
- \( C \) = capacitance to ground per unit length per unit length
- \( K \) = capacitance between turns per unit length

Note here if \( I_k = 0 \) Equations 10.01 and 10.02 reduce to the partial differential equations describing the transmission line. The third equation is the partial differential equation for the interturn capacitive effect. These three equations can be solved simultaneously by differentiating with respect to \( x \) and \( t \) and substituting appropriate terms. The voltage differential equation which results is:

\[ \frac{d^2 v}{dx^2} - LC \frac{d^2 v}{dt^2} + \frac{1}{x} \frac{dv}{dx} - \frac{1}{x^2} \frac{dv}{dt} = 0 \quad (10.04) \]

A similar differential equation for current can be developed. Note here also if \( K = 0 \) the equation reduces to the conventional traveling wave equation.

B. Solution of the Single Layer Coil Differential Equation

The nature of the solution of this differential equation, (10.04), can be determined by examination of the characteristic values or transient solution of the equation. This is most easily done by assuming a solution where the variables are separated. Either of two forms can be used:

\[ v = V e^{\omega t} e^{j2\pi x} \quad (10.05a) \]

or

\[ v = V \cos \omega t \sin 2\pi x \quad (10.05b) \]

Substituting into Equation (10.04) the characteristic equation can be found as:

\[ \omega^2 = L\omega^2 - LC \omega^2 - \frac{1}{x^2} \omega^2 = 0 \quad (10.06) \]

This equation surely states that for every possible oscillation within the coil, \( \omega \), there is a relationship with a spatial factor, \( B \). Solving Equation (10.06) first for \( \omega \) and then for \( B \) obtains:
Thus, for each frequency $u$, there corresponds one $\beta$ and for each $\beta$ there corresponds one frequency $u$. The significance of this characteristic equation evaluation can best be seen by a simple example. If we consider one harmonic; i.e., one value of $u$, we will find that the spatial distribution of the voltage along the winding will be defined by the corresponding value of $\beta$, as shown in Figure 10.03. Thus, the natural frequencies are related to the winding length.

Spatial Distribution of the Voltage Along A Layer Winding

FIGURE 10.03

For this simple spatial wave shown above, the boundary conditions have been chosen to represent an applied voltage at the left and a grounded neutral at the right. A complete solution requires applying boundary conditions to the differential equation to evaluate the magnitude of each harmonic component which will be of the form of Equation (10.05).

No attempt is going to be made here to present a detailed solution to the differential equation, but rather some general comments regarding the solution will be made.

It is possible to use some rather general concepts to evaluate the solution to any differential equation. The first step in many cases is to establish the initial and final values of the solution and then recognize that the transient solution is the solution which must connect the two regions. In this problem it is reasonable to predict the final value of the solution as a linear voltage between the two terminals. This is because the long time or low frequency current flow will be through the inductance branch with negligible capacitive current contribution. Correspondingly we can
consider the equivalent circuit of the winding and deduce the nature of the initial distribution along the winding. Because it takes time for the current to build up in an inductance we can anticipate that the voltage distribution at the first instant will be determined by the capacitive network in Figure 12.04. It is possible to write the differential equation of this network or the equation can be deduced from Equation (12.04) by letting $L = \infty$. In any event the differential equation describing the initial voltage distribution is:

$$\frac{d^2v}{dx^2} + \frac{C}{K} v = 0 \quad (10.08)$$

where:

- $v_0$ is the initial voltage distribution.

We can apply boundary conditions to this equation. Namely, let $v_0 = 1$ at $x = 0$ and let $v_0 = 0$ at $x = L$. These are appropriate initial conditions for a grounded neutral coil. For these initial conditions the solution is:

$$v_0 = \frac{\sinh \alpha (L - x)}{\sinh \alpha} \quad (12.09)$$

where:

$$\alpha = \sqrt{\frac{C}{K}}$$

where:
- $C = \text{total winding capacitance to ground}$
- $K = \text{total winding through capacitance}$

A general plot of this solution can be made for various values of $\alpha$.

![Diagram](image-url)
Now here $\alpha$ is given as $\sqrt{2/\pi}$ and we see that $\alpha$ increases as the interturn capacitance decreases or as the capacitance to ground increases. A typical layer winding with no special shielding to improve the distribution would have a typical $\alpha = 3$ or greater.

In Figure 10.04 we can see the difference between the linear final distribution, the straight line labeled $\alpha = 0$ and the initial voltage distribution, say for instance $\alpha = 3$, must equal the transient solution. This transient solution must be made up of terms of the form of Equation (10.06). In fact, the general form of the solution will be:

$$v = \sum_{n=1}^{\infty} b_n \sin(n \pi x) \cos(n \pi t)$$  \hspace{1cm} (10.10)

The coefficients $b_n$ can be found by fitting the above equation to the difference between the final value and initial value distribution using a Fourier expansion of the areas as shown in Figure 10.05.

![Figure 10.05](image)

**Figure 10.05**

Approximate Curves of Initial Voltage Distribution for Various Values of $\alpha$

In Equation (10.10) the values of $b_n$ can be shown to be:

$$b_n = \frac{\pi}{\alpha}$$  \hspace{1cm} (10.11)

Therefore, the initial portion of the solution is:

$$v = 1 - \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{\alpha} \cos n \pi t$$  \hspace{1cm} (10.12)

The coefficients $b_n$ are found by equating (10.12) to the initial (10.09) value for $t = 0$ and re-writing the equation to obtain:
Now this function can be evaluated using conventional Fourier series techniques to find the coefficients:

\[ b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx \]  

The evaluation of this integral gives:

\[ b_n = \frac{2}{L} \frac{L^2}{n^2 \pi^2} \left( \frac{1}{n^2} - \frac{1}{n^4} \right) \]

Using these values of \( b_n \) in Equation (10.12) the total solution can be written as an infinite series. The values of \( b_n \) decrease for large \( n \) and so only a few terms are required for a complete solution. An example of a plot of spatial harmonics is shown in Figure 10.06.

![Figure 10.06](image_url)

Note here that the sum of the spatial harmonics approximates the difference between the initial and final distribution. These spatial harmonics, \( b_n \sin \frac{n \pi x}{L} \), are each multiplied by a corresponding space harmonic, \( \cos \omega_n t \), to form the space-time solution. That is, each of the spatial harmonics oscillate at its own natural frequency, \( \omega_n \), such that the net solution can be a quite complex distribution of voltage along the winding. A typical example of the voltage along the winding at some instant is shown in Figure 10.07(a). Figure 10.07(b) is more typical of figures presented in papers and texts on this subject and shows the envelope of the maximum voltage at each point on the winding during the total oscillation.
C. Interpretation of the Coil Oscillations Results

The voltage distributions, both initial and actual transient voltage conditions, can be interpreted and discussed relative to system application problems.

First, examination of the voltage gradient along the transformer winding (Figure 10.07(a)) shows the high turn-to-turn voltage that can occur from impulse type terminal voltages. Following the impulse relatively, high frequency oscillations can occur producing high gradients. These high gradients can, in general, occur at both ends of the winding; i.e., both at the line terminal and neutral terminal of the winding. It is also important to observe that the line-to-ground voltage at some point internal to the winding can be higher than the terminal line-to-ground voltage. It is evident that the envelope of overvoltages (Figure 10.07(b)) exceeds the final distribution or equilibrium voltage of the winding and therefore the major insulation of the winding cannot use graded insulation effectively. A graded insulation structure is one where the insulation level is greater at the high voltage end of the transformer and lower near the neutral approximately in proportion to the normal 60 Hz voltage on the winding.

Many modern power transformers do use graded insulation and this is made possible by modifying the transformer design to suppress internal winding oscillations. This can be done through the use of the electrostatic shielding. The electrostatic shield has the effect of decreasing $C$ and increasing
In Equation (10.09) and thereby decreasing $a$. Then as shown in Figure 10.04 the initial voltage distribution can be made more nearly linear.

A number of techniques are employed to alleviate or mitigate the steep voltage profile which would occur in a uniform coil if it were subjected to impulse voltages at its terminals. For example, in shell form winding construction, the winding is arranged in a series of pancakes or discs. Strong pancake-to-pancake as well as turn-to-turn capacitive coupling occurs. For such constructions, the use of static plates at terminals proves very effective in controlling the turn-to-turn voltages resulting during impulse. For core form transformers a number of winding arrangements have been prepared to mitigate this effect. For instance, the concentric layer winding utilizes a sequence of concentric coil layers and line shields to mitigate the turn-to-turn stresses. For disc type windings in core transformers, two techniques have been applied. To begin with, static plates are used to shield the line disc from ground. In addition rib shielding or line shielding is used frequently to increase the through capacitance in the disc sections of the winding near line terminals. Thus, heavily insulated shield conductors are wound over the disc sections near line terminals to increase the capacitive coupling from the line section to the intermediate disc. In addition, the winding construction called the interlaced disc is used to provide multiple turn capacitive coupling. Interlace disc designs greatly increase the effective turn-to-turn capacitance and obviate the need of rib shielding. In all transformer constructions discontinuities in the winding, such as occur in two sections, can lead to voltage stress problems. These sections are sometimes protected by the use of semi-conductor materials with current voltage characteristics tending to limit voltage drop.

From the above formulas and curves it should be evident that the input impedance at a transformer winding, especially under impulse conditions, cannot be easily defined. We can see that the initial impedance is capacitive and the final impedance is inductive.

For a layer winding the initial effective capacitance is:

$$ C_{eff} = \frac{1}{2} \pi K $$  \hspace{1cm} (10.16)

This capacitance can be calculated by observing that the effective capacitance must be equal to:

$$ C_{eff} = -K \frac{\pi x}{\ln^2} \left|_{x = 0} \right. $$  \hspace{1cm} (10.17)

where $K$ is the turn-to-turn capacitance. Substituting Equation (10.09) into Equation (10.17) gives:

$$ C_{eff} = K_0 \ coth \ alpha \cdot \frac{1}{2} \pi K \ coth \ alpha $$  \hspace{1cm} (10.18)
For a coil with a large \( n \) this effective capacitance reduces to equation (10.16) because \( \cosh \alpha > 1 \) for \( \alpha > 0 \). For shielded windings the effective capacitance cannot be defined by the same formula and in general it is difficult to calculate the effective initial capacitance.

In some application studies it is desirable to estimate the effective transformer capacitance, namely on recovery voltage analysis and lightning protection studies. General information is normally used in such studies. In such cases it may be important to include the transformer bushing capacitance as well as the winding capacitance discussed here.

From these introductory calculations we can see where some terms such as transformer effective capacitance came from. It is worth remembering that this impedance may net be capacitive for long-term oscillations but is effective at best for 10's of microseconds.
REFERENCES


6. J. R. Neidor, N. E. Gllow, "Dielectric Tests on Transformers as Influenced by Further Deredu-
cation," paper number 60-60.